Fraïssé-Hrushovski predimensions on nilpotent Lie algebras

D I S S E R T A T I O N

zur Erlangung des akademischen Grades

doctor rerum naturalium
im Fach Mathematik

eingereicht an der
Mathematisch-Wissenschaftlichen Fakultät II
Humboldt-Universität zu Berlin

von
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Florenz, 28.01.80

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eingereicht am: 08.12.2010
Tag der mündlichen Prüfung: 30.05.2011

Abstract

In this work, the so called Fraïssé-Hrushowski amalgamation is applied to nilpotent graded Lie algebras over the p-elements field with p a prime. We are mainly concerned with the *uncollapsed* version of the original process.

The predimension used in the construction is compared with the group theoretical notion of *deficiency*, arising from group Homology.

We also describe in detail the Magnus-Lazard correspondence, to switch between the aforementioned Lie algebras and nilpotent groups of prime exponent. In this context, the Baker-Hausdorff formula allows such groups to be definably interpreted in the corresponding algebras.

Starting from the structures which led to Baudisch' *new uncountably categorical group*, we obtain an ω-stable Lie algebra of nilpotency class 2, as the countable rich Fraïssé limit of a suitable class of finite algebras over $\mathbb{Z}_p$.

We study the theory of this structure in detail: we show its Morley rank is $\omega \cdot 2$ and a complete description of non-forking independence is given, in terms of free amalgams.

In a second part, we develop a new framework for the construction of deficiency-predimensions among graded Lie algebras of nilpotency class higher than 2. This turns out to be considerably harder than the previous case. The nil-3 case in particular has been extensively treated, as the starting point of an inductive procedure.

In this nilpotency class, our main results concern a suitable deficiency function, which behaves for many aspects like a Hrushovski predimension. A related notion of *self-sufficient* extension is given.

We also prove a first amalgamation lemma with respect to self-sufficient embeddings.

Zusammenfassung

In dieser Arbeit wird das Fraïssé-Hrushowskis Amalgamationsverfahren in Zusammenhang mit nilpotenten graduierten Lie Algebren über einem endlichen Körper untersucht.

Die Prädimensionen die in der Konstruktion auftauchen sind mit dem gruppentheoretischen Begriff der *Defizienz* zu vergleichen, welche auf homologische Methoden zurückgeführt werden kann.

Darüber hinaus wird die Magnus-Lazardsche Korrespondenz zwischen den oben genannten Lie Algebren und nilpotenten Gruppen von Primzahl-Exponenten beschrieben. Dabei werden solche Gruppen durch die Baker-Haussdorfsche Formel in den entsprechenden Algebren definierbar interpretiert.

Es wird eine ω-stabile Lie Algebra von Nilpotenzklasse 2 und Morleyrang $\omega \cdot 2$ erhalten, indem man eine *unkollabierte* Version der von Baudisch konstruierten *new uncountably categorical group* betrachtet. Diese wird genau analysiert. Unter anderem wird die Unabhängigkeitsrelation des Nicht-Gabelns durch die Konfiguration des freien Amalgams charakterisiert.

Mittels eines induktiven Ansatzes werden die Grundlagen entwickelt, um neue Prädimensionen für Lie Algebren der Nilpotenzklassen größer als zwei zu schaffen.

Dies erweist sich als wesentlich schwieriger als im Fall 2. Wir konzentrieren uns daher auf die Nilpotenzklasse 3, als Induktionsbasis des oben genannten Prozesses.

In diesem Fall wird die Invariante der Defizienz auf endlich erzeugte Lie Algebren adaptiert. Erstes Hauptergebnis der Arbeit ist der Nachweis dass diese Definition zu einem vernüftigen Begriff selbst-genügender Erweiterungen von Lie Algebren führt und sehr nah einer gewünschten Prädimension im Hrushovskischen Sinn ist.

Wir zeigen – als zweites Hauptergebnis – ein erstes Amalgamationslemma bezüglich selbst-genügender Einbettungen.

Contents

Introduction

The purpose of this work is twofold: on one side we propose a new treatment of the structures which led to Baudisch' *new $\aleph_1$-categorical group* of nilpotency class 2 constructed in [Bau96]. On the other hand we settle a new framework to possibly achieve Groups with similar properties but in higher nilpotency classes. The main efforts involve the nilpotent-3 case.

For what concerns both aspects, the deep contiguity between nilpotent groups of prime exponent and graded Lie algebras over finite fields, let us work within the second kind of structures, which support in addition a linear-algebraic approach. This correspondence is explained in detail in Section 1.4.

The aforemensioned *Baudisch group* arises from a direct translation in combinatorial group-theoretic terms, of the restyled Fraïssé amalgamation technique, which led Hrushovski in [Hru93] to confute Zilber's *structural conjecture* ([Zil84]). We briefly review these facts below, as they form in part the guidelines of the present work.

A definable set of a complete first-order theory is called strongly minimal if its Morley rank and degree are both equal to one. In a strongly minimal structure, the (model-theoretic) algebraic closure yields a *pregeometry*. This allowed for instance Baldwin and Lachlan in [BL71] to reprove Morley's categoricity results by means of a *dimensional* approach, derived by such pregeometries. Strongly minimal structures are in particular $\aleph_1$-categorical and on the contrary, uncountably categorical structures do always "contain" strongly minimal sets as – we might say – building blocks.

For the definition of a (pre)geometry and related notions, the reader is referred to Section 1.1.

The pregeometries attached to the strongly minimal sets definable in a $\aleph_1$-categorical structure, have (after localisation) all isomorphic associated geometries. This local isomorphism type constitutes therefore an invariant of such structures.

Zilber conjectured indeed that each $\aleph_1$-categorical theory T is assigned a geometry according to the following trichotomy (cfr. [Hru93, Goo90]).

1 A disintegrated geometry. No infinite group is definable in T.

2 A nontrivial modular geometry of a vector space. An infinite group is definable in T, but no infinite field does.

3 A non locally modular geometry. T is not one-based and an infinite field is interpretable in T.

The conjecture was disproved by Hrushovski in [Hru93] by means of *new strongly minimal sets*, which have a non-locally modular geometry, but nevertheless do not interpret an infinite field.

These counterexamples rely on a Fraïssé amalgamation procedure (described in Section 1.2), together with a pregeometric machinery, which modifies ordinary embeddings. This allows in particular to control the types of the Fraïssé limit by means of a dimension function: the structures obtained are stable, which is not in general the case for Fraïssé constructions.

To summarise the above process, start – say – from a ternary, order-invariant relation (M, R) and define an integer valued function of the finite parts of the domain M:

$$\delta(A) = |A| - |R(A)| \tag{0.1}$$

where $R(A)$ describes the set of all ternary *links* (a, b, c) with $R(a, b, c)$ – up to permutation – which insist among points of A.

This δ turns out to be a *predimension function* in the sense of Section 1.1.1; there we explain how to derive a pregeometry from any predimension. This yields a dimension function d_M on each $\{R\}$-structure M.

The crucial steps – rather informally – are given below and summarise the approach of [Goo90]. In this paper, Poizat divides the construction into two distinct subsequent steps:

Phase One Define the class $\mathcal{K}$ of all finite $\{R\}$-structures with non-negative predimension. Give a notion of *strong* extensions $A \geqslant B$ in terms of δ and prove $\mathcal{K}$ has the properties of *Hereditarity, Joint Embedding* and *Amalgamation* described in Section 1.2, with respect to $\leqslant$.

The Fraïssé limit K of $(\mathcal{K}, \leqslant)$ obtained is ω-saturated and ω-stable of Morley rank ω and is ultrahomogeneous with respect to $\leqslant$. Types of elements over a set B are discerned in base of their dimension d_K: points which are dependent over B have all finite (unbounded) Morley rank, while transcendent points have all the same type and rank ω. The (forking) geometry of the generic type is the d_K-pregeometry, this is not locally modular. No group diagram is allowed by dimension arguments.

If we restrict the class $\mathcal{K}$ by changing the initial lower bound of δ to a fixed positive integer k, one obtains a Fraïssé limit with a k-transitive, non $k-1$-transitive automorphism group.

Phase Two A proper subclass $\mathcal{K}^\mu$ of $\mathcal{K}$ is defined, for which an $\mathbb{N}$-valued function μ bounds the length of realisations of a family of distinguished *minimal pre-algebraic* extensions. With a more difficult proof, the amalgamation property is true of $\mathcal{K}^\mu$ as well. The theory T^μ of the Fraïssé limit K^μ of $(\mathcal{K}^\mu, \leqslant)$ is strongly minimal.

This second phase is referred to in the literature as the *collapse*, because the finite-rank pre-algebraic types in Phase one, are collapsed to algebraic ones, while as a consequence, the infinite rank type is forced to assume Morley rank 1. The strongly minimal geometry on K^μ *coincides* with the d-pregeometry of K above. [1]

[1] In his PhD thesis [Fer09], Marco Ferreira proves that the geometries of the collapsed structures are isomorphic to the geometry of the regular type in the uncollapsed construction.

In the original paper [Hru93], this bipartite analysis is not present and the amalgamation is carried out directly in the collapsed case. Hrushovski proves the non-interpretability of an infinite group in T^μ as a consequence of *flatness*, a property attributed to the geometry of K^μ. On the other hand Pillay shows in [Pil95], that CM-*trivial* structures do not allow the interpretation of an infinite field: in [Hru93] it is also proved that that the collapsed structure is CM-trivial and that flatness implies CM-triviality.

F. Wagner in [Wag94], provides an axiomatic approach to the above constructions which replaces an explicit predimension argument.

In [Bau96] Baudisch starts from a predimension δ which is very much alike (0.1): it computes the gap between the number of *generators* and *relators* of a suitably *linearised* presentation of groups.

In the perspective of Zilber's trichotomy, he obtains a pure uncountably categorical group of Morley rank 2 with no infinite field interpretable: the associated pregeometry is not locally modular – because the group obtained is connected and non-abelian (cfr. [HP87]) – and its theory is shown to be CM-trivial.

The following result indicates which classes of groups may allow such feature.

Fact ([Bau96, Theorem 2.1])**.** *Assume a connected group G of finite Morley rank does not interpret an infinite field. Then* either *a definable section of G contradicts the Cherlin-Zilber algebraicity Conjecture*[2]*,* or *G is nilpotent.*

In the last case G is the central product of a definable divisible abelian subgroup A and a definable nilpotent subgroup B of G of bounded exponent.

To eventually place ourselves on the "bright side" of Cherlin-Zilber Conjecture, the objects considered in [Bau96] are 2-nilpotent groups of exponent a fixed prime p bigger than 2. Such groups can be reconstructed from the pair of $\mathbb{Z}_p$-vector spaces (G_{ab}, G') – the sections of the lower central series – by means of the linear map $c_G \colon \bigwedge^2 G_{ab} \to G'$, induced by the group commutator in G on the exterior square algebra of G_{ab}. This draws our attention to the pair $(G_{ab}, \ker(c_G))$: step-2 nilpotency yields a 1-1 correspondence of these groups with the structures $(M, N(M))$, where M is a $\mathbb{Z}_p$-vector space and $N(M)$ is a subspace of $\bigwedge^2 M$.

In case of a finitely generated M, one considers

$$\delta(M) = \dim_{\mathbb{Z}_p}(M) - \dim_{\mathbb{Z}_p}(N(M)) \tag{0.2}$$

The Hrushovski amalgamation program described above is carried out in [Bau96] with this δ directly for the Collapsed case, once a suitable native function μ is implicitly given.

In Chapter 2 we recast all the steps leading to (Phase One) of the Hrushovski-Baudisch construction in terms of nilpotent Lie algebras over $\mathbb{Z}_p$.

[2] Infinite simple groups of finite Morley rank are conjectured to be algebraic groups over an algebraically closed field.

In Section 1.4 we present a well-known uniform method to associate a group with a Lie ring, this uses the sections of the lower central series and the group commutator. As a consequence, this procedure becomes particularly effective when dealing with nilpotent groups. If we denote by $\mathfrak{N}_p^c$ the variety of c-nilpotent and exponent p groups, we isolate a class of c-nilpotent graded Lie algebras $\mathfrak{L}_p^c$ over the field $\mathbb{Z}_p$ in order to obtain a *grading* functor gr of $\mathfrak{N}_p^c$ into $\mathfrak{L}_p^c$, which is surjective at the level of objects.

The literature about this subject is founded on the work of Lazard, Magnus [Laz54, Mag40, Mag37] and Witt's [Wit36]. In a *torsion-free* context this phenomenon is also called *Mal'cev Correspondence*: it establishes an equivalence between the categories of torsion-free divisible nilpotent groups and nilpotent Lie $\mathbb{Q}$-algebras (see [Bah78, §6]).

We give two different methods to associate a given Lie algebra L of $\mathfrak{L}_p^c$, a group G of $\mathfrak{N}_p^c$ with $\mathsf{gr}(G) = L$: a group theoretical one, which employes a torsion version of the relationship between free groups and free Lie rings (this is Witt's *Treue Darstellung*) and a more analytical procedure, which uses the Baker-Hausdorff formula. This last approach, although less transparent for higher classes c, has the advantage of establishing a multiplicative group structure $(L, \circ)$ directly on the Lie algebra domain L. This group law will be in fact first-order definable in terms of the ring signature.

The additional requirement $G' = Z(G)$ for the groups G considered in [Bau96], is discussed in Remark 1.4.22. This property, which is preserved by the algebra-group correspondence, will be obtained for $\mathfrak{L}_p^2$-algebras as a consequence of the positive lower bound chosen for the predimension.

In Section 1.4.1 we are concerned with an existing notion of group theoretical *deficiency*, which computes the difference between the generators and the relators of a finitely presented group G. The second integral homology group of G is involved in such a measurement. More precisely, the deficiency of G is always bounded from above, by the difference between the $\mathbb{Z}$-rank of G_{ab} and the minimal number of generators for $H_2(G, \mathbb{Z})$. Following Stammbach and Stallings we derive the correspondent notion of deficiency for groups in the variety $\mathfrak{N}_p^c$ and homology will be taken with coefficients over $\mathbb{Z}_p$.

If we consider a presentation $R \to F \to G$, the so called *Hopf formula* returns $H_2(G)$ as the quotient $R \cap F'/[R, F]$. This term filters in fact the *essential* relators in R, those which actually cause the deficiency to drop.

This filter is basically the same adopted in Chapter 3 for $\mathfrak{L}_p^c$-algebras in order to obtain new kinds of presentations. Despite the strong similarity between the above notions and the relators space we constructed, we encountered this group-homological interpretation only in a very late phase of this work. We decided to include this section as a sort of *a posteriori* motivation.

In the first section of Chapter 2, we start by adapting the deficiency predimension (0.2) to finite objects of $\mathfrak{L}_p^2$.

Any $M = M_1 \oplus M_2$ in $\mathfrak{L}_p^2$, is given by a presentation $R \to L^2(M_1) \to M$ from the free nil-2 Lie algebra $L^2(M_1)$ over M_1. For a subspace A of M_1, the integer $\delta(A)$ (or $\delta_2(A)$ to distinguish from other nilpotency classes) will be defined as $\dim_{\mathbb{Z}_p}(A) - \dim_{\mathbb{Z}_p}(R(A))$.

The relators ideal $R(A) \subseteq L^2(A)$ depends by the ambient relators R and the subspace A.

This function is proved to be a predimension *over* the $\mathbb{Z}_p$-linear closure, as defined in Section 1.1. As a consequence, δ gives rise to a pregeometry on the vector space M_1 whose closure operator extends the linear span. We show directly that this pregeometry is actually a non locally-modular geometry *over* $\mathbb{Z}_p$.

The notion of *self-sufficient* extensions $M \leqslant_2 N$ of $\mathfrak{L}_p^2$-algebras will be given in terms of δ: as usual $\delta(C)$ cannot drop below $\delta(M)$ on spaces C between M and N.

Section 2.2 describes the subclass $\mathcal{K}_2$ of $\mathfrak{L}_p^2$, for which an *asymmetric* amalgamation lemma is shown: we define a free amalgam in $\mathfrak{L}_p^2$, which preserves a positive lower bound of the deficiency, provided a kind of one-point algebraic extensions are suitably avoided. Compared to the correspondent statements in [Bau96], the proofs here are overall simplified, left aside some technicalities (Lemmas 2.2.15 and 2.2.17), which we have to borrow with minor changes from the original text.

As part of this section we find the treatment of *minimal* strong extensions, these will be fundamental for the rank computations in the uncollapsed theory. To this end we prove that chains of minimal extensions commute with free amalgamation.

Asymmetric amalgamation yields a first-order axiom system T^2 for the countable Fraïssé limit of $\mathcal{K}_2$. As it is meant to happen the ω-saturated models of T_2 are exactly the rich structures whose age is $\mathcal{K}_2$. This is Theorem 2.3.1 of Section 2.3, where we also prove ω-stability of T_2 and give a description of the algebraic closure in T_2.

In Section 2.3.1, we explicitly compute the Morley rank of the countable rich model $\mathbb{M}$, which is – as expected[3] – $\omega \cdot 2$.

The reason for this number comes from the $\mathfrak{L}_p^2$-grading $\mathbb{M} = \mathbb{M}_1 \oplus \mathbb{M}_2$ and the locally-free behaviour imposed by the axioms. As our predimension takes its entries among the finite parts of $\mathbb{M}_1$, we first obtain Morley rank ω for this set, by a geometric type analysis à la John B. Goode (cfr. Phase One above). On the other hand, to require a positive deficiency, forces the homogeneous subspaces $\mathbb{M}_1$ and $\mathbb{M}_2$ to be definably $\mathbb{Z}_p$-isomorphic. This doubles the rank. The same happens in the collapsed case and explains the rank 2, there in fact the corresponding set $\mathbb{M}_1$ is strongly minimal.

By applying the aforementioned correspondance we reconstruct a nil-2 group $\mathbb{G}$ which has Morley rank $\omega \cdot 2$. Indeed the whole local construction (amalgamation, self-sufficient embeddings, richness, etc.) can be traced back at the level of groups; cfr. Remark 1.4.22.

A complete description of forking in T^2 follows. This is done in Section 2.3.2 by exhibiting a suitable ternary *independence relation* among sets of the monster model $\mathbb{M}$ which satisfies the axioms of forking in stable theories. This notion of independence reflects both the *geometric* information of the predimension and the *structural* condition imposed by free amalgamation.

In the last section of Chapter 2, we propose a notion of *weak canonical base* for types of self-sufficient tuples over models. This is compared with the properties of *weak*

[3]It is sort of by chance that this value coincides with the rank of the uncollapsed *black field* of Poizat. In that case this factor is artificially obtained by the shape of the predimension, while in ours it closely reflects the structural nil-2 constraint.

elimination of imaginaries and CM-triviality for the *uncollapsed* theory. On this purpose one may also check the notion of *relative* CM-triviality proposed in [BWMP10].

In the third Chapter we study a possible construction of deficiency predimensions in the case of nilpotent Lie algebras from $\mathfrak{L}_p^c$ of class c greater than 2.

The guiding principle here is an inductive approach over the nilpotency class, suggested by the graded shape of a (saturated-homogeneous say) object $\mathbb{M} = \mathbb{M}_1 \oplus \cdots \oplus \mathbb{M}_c$ of $\mathfrak{L}_p^c$.

This corresponds to a presentation $R \to L^c(\mathbb{M}_1) \to \mathbb{M}$ from the free Lie nil-c algebra $L^c(\mathbb{M}_1)$, where the homogeneous ideal R equals $R_2 + \cdots + R_c$ (cfr. Section 1.4). On the other hand, denote by $\mathbb{M}_*$ the *truncation* to $\mathfrak{L}_p^{c-1}$, that is $\mathbb{M}_* = \mathbb{M}/\mathbb{M}_c \simeq \mathbb{M}_1 \oplus \cdots \oplus \mathbb{M}_{c-1}$.

Now assume we have a notion of deficiency δ_{c-1} which locally measures the gap among linear dimensions in $\mathbb{M}_1$ and the numbers of independent relators from $\mathbb{M}_*$ in all possible weights $< c$. Suppose further, such a function behaves like a predimension and yields a dimension function d_{c-1} on $\mathbb{M}_1$. Then we ideally define $\delta_c(A)$ for $A \subseteq \mathbb{M}_1$, as the difference between $d_{c-1}(A)$ and the linear dimension of a new *relators space* $R^c(A)$.

$R^c(A)$ is able to isolate elements of R_c, from Lie products $[\rho, x_1, \ldots, x_{c-k}] \in R_c$, involving relations $\rho \in R_k$ of a lesser weight $k < c$. The definition of $R^c(M)$ is found in Section 3.1.1.

For a fixed prime p and c with[4] $c < p$, in its entirety, this recursive program should produce a sequence of pregeometries $(\mathbb{M}_1, c\ell_i)_{i \leq c}$, each one extending the previous ($c\ell_i \subseteq c\ell_{i+1}$) and all insisting upon the same domain set $\mathbb{M}_1$. Here $c\ell_1$ is the $\mathbb{Z}_p$-linear closure and $c\ell_2$ is the pregeometry obtained from the deficiency δ_2, associated to $\mathfrak{L}_p^2$-algebras.

This aspect motivates the study of extensions among pregeometries and the notion of predimentions *over* a given pregeometry given in Section 1.1.1.

The above operator R^c relies on a *free-lift* functor $\mathsf{fl}\colon \mathfrak{L}_p^{c-1} \to \mathfrak{L}_p^c$ defined in Section 3.1. This is such that $\mathsf{fl}(M)_* = M$ for all M in $\mathfrak{L}_p^{c-1}$ and obey the following universal property: for any other $N \in \mathfrak{L}_p^c$ with $N_* \simeq_{\mathfrak{L}_p^{c-1}} M$, $\mathsf{fl}(M)$ maps uniquely onto N. In other words $\mathsf{fl}(M)$ is the freest possible object in $\mathfrak{L}_p^c$ to have a truncation in $\mathfrak{L}_p^{c-1}$ which is M. We prove in fact that fl is left-adjoint to $_*\colon \mathfrak{L}_p^c \to \mathfrak{L}_p^{c-1}$ in Proposition 3.1.2.

Composed the other way around, the universal property of fl yields, for any algebra M of $\mathfrak{L}_p^c$, the desired *shifted presentation* $R^c(M) \to \mathsf{fl}(M_*) \to M$. The kernel $R^c(M)$ has the properties mentioned above.

This formal strategy is applied, in Section 3.2, in the step from $\mathfrak{L}_p^2$ to $\mathfrak{L}_p^3$. Already in this *induction basis*, major difficulties are encountered in the reproduction of both the Fraïssé procedure and the pregeometric approach.

We define a first deficiency for finitely generated $\mathfrak{L}_p^3$-algebras A, as the difference between $\delta_2(A_*)$ – the $\mathfrak{L}_p^2$-predimension defined in Chapter 2 – and the $\mathbb{Z}_p$-dimension of the space $R^3(A)$ given above.

So defined, this function is unreliable to control deficiencies within a fixed ambient structure M of $\mathfrak{L}_p^3$. That is is because $R^3(A)$ is not in general contained into $R^3(B)$ for extensions $A \subseteq B$ inside M.

[4] The constraint $c < p$ lays in the nature of the Hausdorff series development described in Section 1.4.

This is due to a structural issue intrinsic to the free-lift functor: for extensions $M \subseteq N$ of $\mathfrak{L}_p^2$-structures, the lifted algebra $\mathsf{fl}(M)$ *does not* always embed into $\mathsf{fl}(N)$. In Section 3.1.2 we prove however that if M is a *self-sufficient* $\mathfrak{L}_p^2$-subalgebra of N, then we have a corresponding extension of the lifted $\mathfrak{L}_p^3$-algebras, i.e. $\mathsf{fl}(M) \subseteq \mathsf{fl}(N)$. This crucial result, which influences the whole subsequent construction, is proved by using the so called *Hall's bases* (Definition 1.4.4) of *basic* commutators for free Lie algebras. In fact a similar approach to Hall's *collecting process* in [Hal50] is employed.

Now fixed an $\mathfrak{L}_p^3$-algebra M, we define a more adaptive deficiency $\partial_3^M(A)$, which reads subspaces A of M_1. This is built in terms of the *dimension function* d_2^M – induced by the pregeometry from M_* – and a suitable *monotone* operator $R_M^3(A)$, which returns subspaces of $R^3(M)$ and depends on $\mathsf{fl}(\langle A\rangle)$.

As a consequence of the above embedding result, the functions δ_3 and ∂_3^M do agree on δ_2-strong subalgebras A of M.

This behaviour also suggests the following definition of *strong* $\mathfrak{L}_p^3$-extensions: to write $A \leqslant_3 M$ and say A is self-sufficient in $M \in \mathfrak{L}_p^3$, we require in fact that the truncated structures are self-sufficient with respect to δ_2 ($A_* \leqslant_2 M_*$) and that the auxiliary deficiency ∂_3^M assumes values bigger than $\delta_3(A)$ on all C between A and M_1.

Consequently, we exhibits in Section 3.35 a strong amalgam of $\mathfrak{L}_p^3$-algebras. This is obtained as follows: start with a strong configuration like $A\, {}_3\!\geqslant B \leqslant_3 C$, then take the truncated preamalgam $A_*\, {}_2\!\geqslant B_* \leqslant_2 C_*$ and obtain, with the results in Chapter 2, a free $\mathfrak{L}_p^2$-amalgam D_* of A_* and B_* over C_*.

This yields strong $\mathfrak{L}_p^2$-inclusions $A_* \leqslant_2 D_*\, {}_2\!\geqslant C_*$. Now take the free-lift $\mathsf{fl}(D_*)$ and by virtue of the aforementioned fact, obtain the embeddings $\mathsf{fl}(A_*) \subseteq \mathsf{fl}(D_*) \supseteq \mathsf{fl}(C_*)$.

Since A and C are isomorphic to the quotients $\mathsf{fl}(A)/R^3(A)$ and $\mathsf{fl}(C)/R^3(C)$, the $\mathfrak{L}_p^3$-algebra $D := \mathsf{fl}(D_*)/(R^3(A) + R^3(C))$, amalgamates A and C over B and we show $A \leqslant_3 D\, {}_3\!\geqslant C$ in Lemma 3.2.10.

With a modified procedure we were actually able to prove the *asymmetric* version of the above result: from $A \supseteq B \leqslant_3 C$, we obtain $A \leqslant_3 D \supseteq C$. As shown in Chapter 2 in fact, asymmetric amalgamation is indispensable to approximate richness in a possible axiomatisation of the Fraïssé limit.

A further remark, independent of previous issues, settle at this point the following – and more critical – problem: *to decide whether $R_M^3(A) \cap R_M^3(B)$ equals $R_M^3(A \cap B)$, for given subspaces A and B of M_1.*

The answer is negative in general and two main obstructions follow thereafter:

- we prove with examples, that ∂_3^M (and δ_3) is not in general submodular.
- We cannot prove the strong $\mathfrak{L}_p^3$-embedding $\leqslant_3$ is transitive, nor find a transitive notion related to $\leqslant_3$ [5].

[5] there is a standard way to *force* transitivity via a local "cut lemma" (cfr. Lemma 2.1.12) definition of strongness: in our case one should define A *is strong* in M if for any finite part U of M_1, $\delta_3(A_1 \cap U) \leq \delta_3(U)$. This definition however does not comply with the amalgamation in $\mathfrak{L}_p^3$ described in Lemma 3.2.10.

The first makes void the proof-strategies adopted in Chapter 2. Submodularity is in fact on one hand the key property to turn a deficiency-like function into a predimension, on the other, it ensures that free amalgamation preserves the same lower bound for the deficiency, of the amalgamated structures.

The efforts of Section 3.2.2 goes in the direction of finding *local* conditions to force a modular behaviour of R^3_M and hence be able to use submodularity of ∂_3 *just where we need it*.

This is strongly connected to the relationship between δ_3 and ∂_3. In this section we prove indeed that they are uniformly comparable, namely in the direction $\partial_3^M(A) \leq \delta_3(A)$ for any finite algebra A of $\mathfrak{L}^3_p$.

In accordance to this and the above amalgamation process, we define a class $\mathcal{K}_3$ of $\mathfrak{L}^3_p$-algebras M with M_* in $\mathcal{K}_2$ for which δ_3 is non-negative on the finite subalgebras of M. By the above, we can use indifferently δ_3 or ∂_3^M to test whether M is in $\mathcal{K}_3$.

We indicate $\mathcal{K}_3$ as a possible candidate to represent the age of the desired rich $\mathfrak{L}^3_p$-algebra, although we couldn't prove the amalgamation property for $\mathcal{K}_3$.

The exclusive treatment of the uncollapsed case in this work is also motivated by a later project of Baudisch', *The Additive Collapse* ([Bau09]). Here an ω-stable theory T is considered, which expands the theory of vector spaces over the finite field $\mathbb{Z}_p$. A pregeometry is assigned on the models of T and a notion of *strong embedding* between subspaces is given, which both influence the elementary type of the saturated monster $\mathbb{K}$ of T. Further properties are required of T, which capture the essential features of the *uncollapsed* infinite rank versions of the known amalgamation examples.

After *prealgebraic codes* and the aforementioned bound-function μ are chosen, the collapsed structure $\mathbb{K}^\mu$ of finite rank, is constructed directly *inside* $\mathbb{K}$.

This new procedure is meant to unify under a common frame, the Red fields [BMPZ07], the new uncountably categorical group and the fusion over a vector space [BMPZ06].

Should suitable stable rich $\mathfrak{L}^c_p$-algebras ($c > 2$) be constructed with the methods described in the present work, then the additive collapse process would give finite rank nilpotent Lie algebras or groups, with underlying Hrushovski geometries.

1 Basic Facts and Definitions

The notation we use is overall standard. Through the whole work, maps are sometimes – especially in Chapter 3 – applied *on the right* of their arguments. When this happens, composition of applications follows the natural *left-to-right* notation.

1.1 Combinatorial Pregeometries

If M is a set, $\wp(M)$ denotes its powerset. To denote unions of sets, juxtaposition will almost everywhere replace the symbol $\cup$ in the sequel, so that AB will mean $A \cup B$ and Ab will be $A \cup \{b\}$ for all sets A, B and elements b, of M.

Definition 1.1.1. A *pregeometry* (M, cl) is a set M endowed with a closure operator $\mathit{cl}\colon \wp(M) \to \wp(M)$ on M, which satisfies the *Steiniz exchange property.* This means the following properties are required of cl:

cl1) $A \subseteq \mathit{cl}(A)$ for all $A \in \wp(M)$

cl2) $\mathit{cl} \circ \mathit{cl} = \mathit{cl}$

fin) $\mathit{cl}(A)$ is the union of all $\mathit{cl}(B)$, where the B's range over the finite parts of A.

ex) For all $a, b \in M$ and all $A \in \wp(M)$, when $a \in \mathit{cl}(Ab) \setminus \mathit{cl}(A)$, then $b \in \mathit{cl}(Aa)$

If in addition $\mathit{cl}(\varnothing) = \varnothing$ and $\mathit{cl}(a) := \mathit{cl}(\{a\}) = \{a\}$ for all singletons $a \in M$, we say that (M, cl) is a *geometry*.

Note that (fin) alone implies monotonicity of the closure operator cl. A *closed set* of M is defined, as usual, as a fixed point of cl.

From a pregeometry (M, cl) we obtain a geometry (M_*, cl_*) if we define M_* as $(M \setminus \mathit{cl}(\varnothing))/_\sim$ for $a \sim b \iff \mathit{cl}(a) = \mathit{cl}(b)$, and $\mathit{cl}_*(A/_\sim)$ to be $\mathit{cl}(A)/_\sim$. This procedure is exactly the way a projective space is obtained out of a vector space: each line is identified to a point.

If (M, cl) is a pregeometry and B a subset of M we define its *localisation at* B as the pregeometry (M, cl_B) given by $\mathit{cl}_B(U) := \mathit{cl}(BU)$ for all subsets $U \subseteq M$.

We say that the subset A of M is *independent over* B (or B-*independent*) if $a \notin \mathit{cl}_B(A \setminus \{a\})$ for all $a \in A$.

We say that a subset C of $A \subseteq M$ is a *base for* A *over* B, if it is independent over B and $A \subseteq \mathit{cl}_B(C)$. The definition of an independent set or of a base of a set are obtained if we put B above to be the empty set.

By the exchange property, given any set A, a maximal B-independent subset of A is a base over B. Moreover all bases have the same cardinality[1], which is defined as the *dimension of A over B* and denoted with $\dim(A/B)$. This (ordinal) number satisfies the following additivity property

$$\dim(AB) = \dim(B) + \dim(A/B) \tag{1.1}$$

for any sets A and B.

We may also say that a set D is independent of C over B if $\dim(D/B) = \dim(D/CB)$.

Definition 1.1.2. A pregeometry (M, cl) is *trivial* or *disintegrated* if for any sets $A, B \subseteq M$ we have $cl(AB) = cl(A) \cup cl(B)$.

We say that a pregeometry (M, cl) is *modular* if for all closed sets A and B, we have $\dim(A/B) = \dim(A/A \cap B)$.

A pregeometry (M, cl) is *locally modular* if the above equality is true whenever $\dim(A \cap B) > 0$ or equivalently if $(M, cl_{\{a\}})$ is modular for all $a \in M$.

Remark that a trivial pregeometry is always modular, and that a modular geometry is also locally modular. Moreover a pregeometry is modular exactly if any closed set A is independent of any closed B over their intersection and also iff the following equality holds on finite-dimensional closed sets A, B

$$\dim(AB) + \dim(A \cap B) = \dim(A) + dim(B). \tag{1.2}$$

It is routine to mention the following examples:

- A vector space V over a field $\boldsymbol{k}$ is a pregeometry if we set $cl(A) = \langle A \rangle_{\boldsymbol{k}}$, the $\boldsymbol{k}$-linear span of a subset A in V. This is a non-trivial modular pregeometry.
- If $\mathbb{A}$ is an affine space with underlying vector space V, the affine closure turns $\mathbb{A}$ into a non-modular, locally modular pregeometry.
- Algebraic closure in an algebraically closed field (of large enough transcendence degree) gives rise to a non-locally modular pregeometry.

1.1.1 Predimensions and associated Pregeometry Extensions

We denote by $[M]$ the set of the finite parts of M.

Definition 1.1.3. Assume cl and cl_0 are closure operators which both turn M into a pregeometry. We say that cl *extends* cl_0 if for all $A \subseteq M$ we have $cl_0(A) \subseteq cl(A)$.

We say that (M, cl) is a *geometry over* cl_0, if cl extends cl_0, if $cl(\varnothing) = cl_0(\varnothing)$ and if $cl(a) = cl_0(a)$ for all $a \in M$. In the case cl_0 is the identical closure ($cl_0(A) = A$, for all $A \subseteq M$) (M, cl) is called a *geometry*.

[1]Exchange property is essentially needed to prove that *finite* bases have all the same size.

If $\mathcal{cl}$ extends $\mathcal{cl}_0$ and dim, $\dim_0$ denote the associated dimensions, then for each $A \in [M]$ we have clearly $\dim_0(A) \geq \dim(A)$, moreover

$$\dim(\mathcal{cl}_0(A)) \leq \dim(\mathcal{cl}(A)) = \dim(A) \quad \text{and} \quad \dim(A) \leq \dim(\mathcal{cl}_0(A)).$$

In particular $\dim(A) = \dim(\mathcal{cl}_0(A))$, that is, dim is determined by its value on $\mathcal{cl}_0$-closed sets.

Let now $(M, \mathcal{cl})$ be a pregeometry, we denote the set of *finitely generated $\mathcal{cl}$-closed parts* of M by

$$[M]_{\mathcal{cl}} = \{B \subseteq M \mid B \text{ is } \mathcal{cl}\text{-closed with } \dim(B) \text{ finite}\}.$$

Definition 1.1.4. We call a map $\delta\colon [M]_{\mathcal{cl}} \to \mathbb{N}$ a *$\mathcal{cl}$-predimension* (or a *predimension over $\mathcal{cl}$*) on M if the following holds:

$$\delta(\mathcal{cl}(\varnothing)) = 0 \quad \text{and} \quad \delta(\mathcal{cl}(a)) \leq 1 \qquad \text{(normalization)}$$

$$\delta(\mathcal{cl}(UV)) \leq \delta(U) + \delta(V) - \delta(U \cap V) \qquad (1.3)$$

for all $a \in M$ and $U, V \in [M]_{\mathcal{cl}}$. Compared to (1.2), property (1.3) above is referred to as *submodularity.*

A *predimension* on M is, by definition, a $\mathcal{cl}$-predimension where $\mathcal{cl}$ is the identical closure on M.

A predimension d on M which is *monotone*, that is $d(B) \leq d(A)$ for all finite $B \subseteq A$ in $[M]$ is called a *dimension function* on M.

Assume δ is a $\mathcal{cl}$-predimension on M and set, for all A in $[M]$

$$d(A) := \min(\delta(C) \mid C \in [M]_{\mathcal{cl}},\ C \supseteq A). \qquad (1.4)$$

With the above definition we still have $d(\varnothing) = 0$ and $d(a) \leq 1$ for all singleton a. Moreover for finite A, B in M let us choose $\mathcal{cl}$-closed oversets $A' \supseteq A$ and $B' \supseteq B$ with $d(A) = \delta(A')$ and $d(B) = \delta(B')$. Since closed sets are closed under intersection, we have

$$d(AB) + d(A \cap B) \leq \delta(\mathcal{cl}(A'B')) + \delta(A' \cap B') \leq d(A) + d(B),$$

that is d is a dimension function on M after Definition 1.1.4 above, and is called *the* dimension function *associated to* δ. Also note that, in the definition of d, is crucial to require δ to be non-negative.

The next lemma shows that d is actually the dimension associated to a prescribed pregeometry.

Lemma 1.1.5. *Assume d is the dimension function associated to a $\mathcal{cl}$-predimension δ on the set M via (1.4).*

For any $A \in [M]$ define $\mathcal{cl}_d(A)$ to be the set of all b of M such that

$$d(Ab) = d(A).$$

If we define $\mathcal{cl}_d$ on arbitrary sets in the natural way, that is by putting $\mathcal{cl}_d(A) = \bigcup\{\mathcal{cl}_d(F) \mid F \in [A]\}$, then $(M, \mathcal{cl}_d)$ is a pregeometry which extends $(M, \mathcal{cl})$ and has dimension d.

Proof. It is enough to show properties (cl2) and (ex) holds for $\mathcal{cl}_d$ over finite sets, while (cl1) is clear.

For (cl2) assume that a finite set B is contained in $\mathcal{cl}_d(A)$ for some finite set $A \subseteq M$. That is $d(Ab) = d(A)$ for all b in B. By induction on the cardinality of B, using submodularity (1.3), it follows $d(BA) = d(A)$.

We need to show that if an element a is in $\mathcal{cl}_d(B)$ then it is in $\mathcal{cl}_d(A)$. Applying submodularity we have $d(aBA) \leq d(aB) + d(BA) - d(B) = d(BA)$.

We can conclude $d(aA) \leq d(aBA) \leq d(A)$ as desired. This gives $\mathcal{cl}_d(A) \supseteq \mathcal{cl}_d(\mathcal{cl}_d(A))$ and (cl2) follows.

To obtain the exchange property (ex), observe first that, as as a dimension-function, d satisfies $d(Ab) \leq d(A) + 1$, for any finite $A \subseteq M$.

Assume $a \in \mathcal{cl}_d(A\,b) \setminus \mathcal{cl}_d(A)$, this means $d(A) < d(aA) \leq d(aAb) = d(Ab) \leq d(A)+1$. This forces b to be in the closure of Aa.

That $\mathcal{cl}_d$ extends $\mathcal{cl}$ is readily seen, as $d(Ab) = d(A)$ for all $b \in \mathcal{cl}(A)$ by definition (1.4), for all $A \in [M]$.

And it is trivial to verify that $d(S)$ is the $\mathcal{cl}_d$-dimension of S, for every set S in M.

□

1.2 Fraïssé Limits

We refer to a *Fraïssé amalgamation construction* in general as the technique introduced by Roland Fraïssé in [Fra54] to recover universal-homogeneous structures from a prescribed class of finite ones. We follow the treatment of Ziegler-Tent [ZT10] for the countable setting. One may also check [BS69] for similar constructions in arbitrary cardinality.

The original results of [Fra54] are stated in a relational language, but they remain true in the following wider context.

Let $\mathscr{L}$ be a countable language; if we say embedding below, we mean $\mathscr{L}$-embedding. Given an $\mathscr{L}$-structure M we define *the age of* M, denoted âge(M) to be the class of all finitely generated $\mathscr{L}$-structures which are isomorphic to a substructure of M.

Given a class $\mathcal{K}$ of finitely generated $\mathscr{L}$-structures which is closed under isomorphisms, we define the following properties for $\mathcal{K}$:

(HP) For each object A in $\mathcal{K}$ and substructure $B \subseteq A$, we have âge$(B) \subseteq \mathcal{K}$ (Hereditary Property).

(JEP) Given any two objects B and C, there exists A in $\mathcal{K}$ which embeds B and C (Joint Embedding Property).

(AP) For all $D \in \mathcal{K}$ and embeddings $\beta\colon D \hookrightarrow B$ and $\gamma\colon D \hookrightarrow C$ there exists an object A of $\mathcal{K}$ and embeddings $b\colon B \hookrightarrow A$ and $c\colon C \hookrightarrow A$ such that $\beta b = \gamma c$ (Amalgamation Property).

Remark that (JEP) does not follow in general by Amalgamation, as provided by the class of (finite) fields without a specified characteristic.

Define $\widetilde{\mathcal{K}}$ to be the class of all $\mathscr{L}$-structures whose age is contained in $\mathcal{K}$.

For a given M, âge(M) satisfies of course (HP) and (JEP), while for âge(M) to have (AP) it is necessary to require M is *strongly homogeneous*. How much, is explained by the next definition and facts.

Definition 1.2.1. An $\mathscr{L}$-structure M is said $\mathcal{K}$*-rich* if âge$(M) = \mathcal{K}$ and for any embedding $\beta\colon B \hookrightarrow A$ of $\mathcal{K}$-objects A and B, if b is an embedding of B into M then there exists an embedding $a\colon A \hookrightarrow M$ such that $\beta a = b$.

For the proof of the following result we refer to [ZT10, Theorem 13.4].

Fact 1.2.2 (Fraïssé Limits)**.** *Let $\mathcal{K}$ be a denumerable class of $\mathscr{L}$-structures for a countable language $\mathscr{L}$ which is closed under isomorphism, then*

(i) *there exists a countable $\mathcal{K}$-rich $\mathscr{L}$-structure M in $\widetilde{\mathcal{K}}$ iff $\mathcal{K}$ satisfies (HP), (JEP) and (AP).*

(ii) *Any two countable $\mathcal{K}$-rich structures are isomorphic. More generally, any two $\mathcal{K}$-rich structures are $\mathscr{L}_{\infty,\omega}$-equivalent, that is, they can be matched up by an infinite back and forth correspondence.*

The isomorphism type of the countable rich structure is called the *Fraïssé limit* of the class $\mathcal{K}$.

The class $\widetilde{\mathcal{K}}$ is not in general elementary. The classical first examples of this construction are the class of finite linear orders, which has $(\mathbb{Q}, <)$ as Fraïssé limit and the class of finite undirected graphs, of which the Random Graph is the limit.

Let now a denumerable class $\mathcal{K}$ be given, of finitely generated $\mathscr{L}$-structures, for a countable language $\mathscr{L}$. Assume $\leqslant$ is a binary relation among objects of $\mathcal{K}$, which is contained in the $\mathscr{L}$-embedding relation and which is invariant under $\mathscr{L}$-isomorphisms.

Remark 1.2.3. Suppose $(\mathcal{K}, \leqslant)$ is a partial order and the properties (JEP) and (AP) are true of $\mathcal{K}$ with $\leqslant$ replacing $\mathscr{L}$-embeddings, while (HP) holds in the original fashion. Then Fact 1.2.2 applies to this situation: there exists a countable structure $\mathbb{K}$ in $\widetilde{\mathcal{K}}$, which is rich with respect to $\leqslant$. With this we mean just β, b and a are to be replaced with $\leqslant$-embeddings in Definition 1.2.1.

We may write in this case that $\mathbb{K}$ is the Fraïssé limit *of* $(\mathcal{K}, \leqslant)$.

Hrushovski's construction relies on the above modification. As described in the Introduction, the "ab Initio" example substitutes embeddings among relations with *predimensionally strong* embeddings.

In Secton 2.2, we describe a similar approach: Lie algebra embeddings are replaced by a suitable stronger notion.

1.3 A few Notions from Stability

The facts from stability theory we use are quite basic. In the case of the uncollapsed nil-2 Lie algebra we construct in Section 2.3, the theory obtained is ω-stable. Only properties of such theories will therefore be needed. For the concepts and the definitions of this section, we essentially follow [Zie98] or [ZT10].

By a *totally transcendental theory* we mean a theory in which each formula $\varphi(\bar{x})$ in n variables has ordinal Morley rank, for all $n < \omega$. By Fact 1.3.2 below this is equivalent to require every 1-formula to have ordinal Morley rank or to require the formula $x = x$ to have such a rank.

Fact. *An ω-stable theory T is totally transcendental (short t.t.). Moreover the two notion coincide if the language of T is countable.*

The facts recalled in the rest of the section, if not otherwise specified, concern a fixed large saturated monster $\mathfrak{C}$ of a totally transcendental theory T. *Small* sets are subsets of $\mathfrak{C}$ whose cardinality is less than $|\mathfrak{C}|$ and *models* are small elementary substructures of $\mathfrak{C}$.

We assume Morley rank and degree are defined on partial types p in T over parameters from $\mathfrak{C}$ and write respectively $\mathrm{MR}(p)$ and $\mathrm{Md}(p)$. We also denote by $\mathrm{MRd}(p)$ the ordered pair $(\mathrm{MR}(p), \mathrm{Md}(p))$.

For a formula $\varphi(\bar{x})$, $\mathrm{MRd}(\varphi(\bar{x}))$ stands for $\mathrm{MRd}(\{\varphi(\bar{x})\})$ while $\mathrm{MR}(\bar{a}/B)$ will be the Morley rank of $\mathrm{tp}(\bar{a}/B)$, for any tuple $\bar{a}$ and small subset set B of $\mathfrak{C}$.

Remark 1.3.1. Morley rank is *continuous*, that is for any complete type p, $\mathrm{MR}(p)$ is the rank of a formula ϕ in p, and for any complete type $q \ni \phi$, $\mathrm{MR}(q) \leq \mathrm{MR}(\phi)$. Moreover for any formula $\psi(\bar{x})$ we have dually

$$\mathrm{MR}(\psi(\bar{x})) = \max(\mathrm{MR}(p) \mid p \in S_{\bar{x}}(A), \psi \text{ is over } A \text{ and } \psi \in p).$$

Both statements of the following fact will be used in Section 2.3.1 further below. The first is an easy exercise on rank computation under algebraicity, the second is due to a result of Erimbetov ([Eri75]), in the formulation of which we follow [Zie97]. The *product* of two ordinals α and β, denoted by $\alpha \cdot \beta$, is defined as the order type of the (inverted) lexicographic order on $\alpha \times \beta$.

Fact 1.3.2. *Let $f\colon \mathscr{D} \to \mathscr{E}$ be a definable map between definable (possibly with parameters) classes of the monster model of an arbitrary theory.*

1. *If f is finite-to-one and onto $\mathscr{E}$, then $\mathrm{MR}(\mathscr{D}) = \mathrm{MR}(\mathscr{E})$.*
2. *If $\mathscr{E}$ has Morley rank β and the Morley rank of all fibres $f^{-1}(e)$ is bounded by an ordinal $\alpha > 0$, the Morley rank of $\mathscr{D}$ is bounded by $\alpha \cdot (\beta + 1)$.*

We assume a notion of *non-forking* extension of types is given (through *dividing*). In a totally transcendental setting, non-forking is expressed in terms of Morley rank. For tuples $\bar{a}$ and small subsets A, B in $\mathfrak{C}$ we write

$$\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}^{f}_{B} A \iff \mathrm{MR}(\bar{a}/B) = \mathrm{MR}(\bar{a}/AB). \tag{1.5}$$

to say that $\mathrm{tp}(\bar{a}/AB)$ *does not fork over* B.

The *Lascar rank* ([Las76]) on complete types p of a stable theory, denoted by $U(p)$ is the smallest *connected* notion of rank on complete types, whose gap on type extensions witness forking (see also [HH84] or and [Bue96, §6]). This means, if q extends p, q is a forking extension of p iff $U(q) < U(p)$.

Moreover connectedness means, that $U(p) = \alpha$ and $\beta \leq \alpha$ implies the existence of a complete type $q \supseteq p$ with $U(q) = \beta$.[2]

In particular we have $U(p) \leq \mathrm{MR}(p)$ on all complete types p of T.

The strength of Lascar rank lays in its additive property. We refer to the book of Buechler cited, for the definition of the *commutative sum* $\alpha \oplus \beta$ of two ordinal numbers α, β.

Fact 1.3.3 ([Las76, Theorem 8])**.** *In a superstable theory T, for all tuples $\bar{a}, \bar{b}$ and sets B, we have*

$$U(\bar{a}/B\bar{b}) + U(\bar{b}/B) \leq U(\bar{a}\bar{b}/B) \leq U(\bar{a}/B\bar{b}) \oplus U(\bar{b}/B).$$

Moreover since the ordinal sum $+$ and $\oplus$ coincide on finite ordinals, when $U(\bar{a}/B\bar{b})$ and $U(\bar{b}/B)$ are both finite, we have

$$U(\bar{a}\bar{b}/B) = U(\bar{a}/B\bar{b}) + U(\bar{b}/B).$$

This additive behaviour resembles additivity of Morley rank in strongly minimal sets and will turn out very useful when computing the rank of the theory T_2 in Section 2.3.

Unfortunately the two notions of rank introduced so far do not in general coincide on complete types even in an ω-stable context. In [Bue96, §6 and §7] one finds an extensive account of examples and conditions under which these ranks do or do not coincide. Among the affirmative cases we find, for instance, the uncountably categorical theories.

Rank computations in Section 2.3.1 involve the following very special instance, which we prove below

Lemma 1.3.4. *Let $\mathfrak{X}$ be a family of complete isolated types in T, over finite sets of parameters.*

Suppose further that for any type $p \in \mathfrak{X}$ and each finite set C containing the parameters of p, any complete extension of p over C lays again in $\mathfrak{X}$.

Then Morley rank and U-rank agree on $\mathfrak{X}$.

Proof. Let $p \in S_{\bar{x}}(A)$ be a type in $\mathfrak{X}$ for a finite set A, and assume that $MR(p) \geq \alpha$ for some ordinal number α, we show by induction on α, that $U(p) \geq \alpha$. Let the statement by true of types from $\mathfrak{X}$ for all ordinals $\alpha < \kappa$. If κ is a limit ordinal, then by the definition of ranks it follows $U(p) \geq \kappa$.

[2] Morley rank is connected with respect to *formulas* but not on complete types.

Let κ be $\alpha+1$ for some ordinal α, and $\mathrm{MR}(p) \geq \alpha+1$. Since p is isolated, there is a formula $\varphi(\bar{x})$ over A which implies p, hence $\mathrm{MR}(\varphi) \geq \alpha+1$. Since T is t.t., let $\psi(\bar{x})$ be a formula over some finite $C \supseteq A$ implying φ with $\mathrm{MR}(\psi)=\alpha$. Choose a type q in $S_{\bar{x}}(C)$ generic in ψ, then we have $\mathrm{MR}(q)=\alpha$, q implies φ and hence q is a forking extension of p. This yelds $q \in \mathfrak{X}$ and since $\mathrm{MR}(q) \geq \alpha$, by induction, $U(q) \geq \alpha$, this means exactly $U(p) \geq \alpha+1$.

We actually showed that the assumptions force Morley rank to be connected on $\mathfrak{X}$.

□

We will also need the characterisation of forking in terms of a *notion of independence*: a "stable version" of Kim-Pillay results for simple theories.

With this respect, we follow the approach of [HH84, Theorem 5.8] and Ziegler and Tent in ([ZT10, Theorem 36.10]).

The last authors seem to exhibit an overall *shortest* list of properties for a distinguished class of type extensions to coincide with the non-forking relation. We stick however to the equivalent formulation in terms of an independence relation among *sets* rather than types.

Fact 1.3.5. *Assume a complete theory T is endowed – for each $n<\omega$ – with a ternary relation $\bar{x} \mathop{\smile\!\!\!\!\!\!\perp}_X Y$ between tuples $\bar{x}$ of length n and pairs of (small) sets X, Y of T, which is invariant under* $\mathrm{Aut}(\mathfrak{C})$. *Then T is stable if an only if $\mathop{\smile\!\!\!\!\!\!\perp}$ satisfies:*

(Local Character) *there is a cardinal κ such that for all tuple $\bar{a}$ and set C, there is $C_0 \subseteq C$ of cardinality at most κ such that $\bar{x} \mathop{\smile\!\!\!\!\!\!\perp}_{C_0} C$.*

(Boundedness) *There is a cardinal μ such that for all $A \supseteq B$ and any tuple $\bar{a}$, there are at most μ $\mathrm{Aut}_A(\mathfrak{C})$-orbits among tuples $\bar{a}'$ with $\bar{a}' \mathop{\smile\!\!\!\!\!\!\perp}_B A$ and $\bar{a} \equiv_B \bar{a}'$.*

If in addition $\mathop{\smile\!\!\!\!\!\!\perp}$ satisfies, for all sets $A \supseteq B \supseteq C$:

(Transitivity) *for any tuple $\bar{a}$, from $\bar{a} \mathop{\smile\!\!\!\!\!\!\perp}_C B$ and $\bar{a} \mathop{\smile\!\!\!\!\!\!\perp}_B A$, follows $\bar{a} \mathop{\smile\!\!\!\!\!\!\perp}_C A$.*

(Monotony) *For all $\bar{a}$, $\bar{a} \mathop{\smile\!\!\!\!\!\!\perp}_C A$ implies $\bar{a} \mathop{\smile\!\!\!\!\!\!\perp}_C B$.*

(Existence) *for any $\bar{a}$ and $B \supseteq C$ there exists a tuple $\bar{a}'$ with $\bar{a}' \equiv_C \bar{a}$ such that $\bar{a}' \mathop{\smile\!\!\!\!\!\!\perp}_C B$.*

then $\mathop{\smile\!\!\!\!\!\!\perp}$ coincides with non-forking independence, that is $\bar{a} \mathop{\smile\!\!\!\!\!\!\perp}_B A$ holds, exactly when $\mathrm{tp}(\bar{a}/AB)$ does not fork over B.

Of course properties above specialise to the case t.t. theories, i.e. Local Character becomes *finite* Local Character and a *finite* instance of Boundedness property is satisfied.

On the contrary, finite local character and finite boundedness of a notion of independence in a small theory imply ω-stability.

Remark 1.3.6. Stable forking independence satisfies in addition:

$$\forall \bar{a}, \bar{b}, B, \quad \bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{B} \bar{b} \iff \bar{b} \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{B} \bar{a} \qquad \text{(Symmetry)}$$

$$\forall \bar{a}, B, \quad \bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{B} \bar{a} \Rightarrow \bar{a} \in \operatorname{acl}(B) \qquad \text{(Irreflexivity)}$$

$$\forall \bar{a},\ C \subseteq \operatorname{acl}(A, B) \text{ and } \bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{B} A \quad \Rightarrow \bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{B} C \qquad \text{(Algebraicity)}$$

$$\forall \bar{a}, A \supseteq B \supseteq C, \quad \bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{C} A \Rightarrow \bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{B} A \qquad \text{(Base Monotonicity)}$$

For a comprehensive account on the possible axiomatic choices for a notion of independence we refer to [Adl07].

In the last section of Chapter 2, we prove some results around weak elimination of imaginaries and also draw a strategy toward a proof of CM-triviality for our uncollapsed structure.

We recall below some essential facts about these notions, following [Zie98] and [CF04].

Definition 1.3.7. A theory T has *weak elimination of imaginaries* (WEI) if for every imaginary element e, in M^{eq} for any model M of T, there is a real tuple $\bar{c}$ such that e is definable over $\bar{c}$ and $\bar{c}$ is algebraic over e, that is

$$e \in \operatorname{dcl}^{eq}(\bar{c}) \quad \text{and} \quad \bar{c} \in \operatorname{acl}^{eq}(e).$$

Imaginary elements are used essentially to deal with canonical bases of types and definable sets.

In our t.t. theory T for a complete stationary type $p = p(\bar{x})$ ($\mathrm{Md}(p) = 1$) over a set A, the *canonical base* of p is the definable closure $Cb(p)$ of the – at most $|T|$-many – canonical parameters of the p-definition formulas $d_p x \varphi(x, y)$ ([Zie98, p.29]) as $\varphi(x, y)$ ranges over the language of T. In our context $Cb(p)$ is the definable closure of a finite sequence of imaginaries.

$Cb(p)$ lays a priori in $\mathfrak{C}^{eq}$ and is point-wise fixed by exactly those automorphism σ of $\mathfrak{C}$ for which p and p^{σ} have the same global non-forking extension. Therefore if $\boldsymbol{p}$ is a global type, $\boldsymbol{p}$ is fixed by exactly the automorphisms which fixes $Cb(\boldsymbol{p})$ point-wise. For a global type $\boldsymbol{p}$ and a set A of parameters we will also need the following renown property of canonical bases:

Fact 1.3.8 ([Zie98, Theorem 4.2])**.**

(1.) $\boldsymbol{p}$ *does not fork over* A *iff* $Cb(\boldsymbol{p}) \subseteq \operatorname{acl}^{eq}(A)$

(2.) $\boldsymbol{p}$ *is the unique non-forking extension of the (stationary) type* $\boldsymbol{p}\restriction_A$ *iff* $Cb(\boldsymbol{p}) \subseteq \operatorname{dcl}^{eq}(A)$.

We will write $Cb(\bar{a}/B)$ to denote $Cb(\operatorname{tp}(\bar{a}/B))$, provided $\operatorname{tp}(\bar{a}/B)$ is stationary.

The following result which may be derived from [CF04, Proposition 2.5] will be also mentioned in Section 2.3.3. It is a statement about the existence of *weak* canonical bases for types over models. For ease of reference, we adapt the proof to the total transcendental setting.

Lemma 1.3.9. *T has* (WEI) *if and only if for any (small) model $M \preccurlyeq \mathfrak{C}$, and any type $p \in S(M)$, there exists a* real *tuple $\bar{c}$ in M such that:*

(i) *the pointwise stabiliser of $\bar{c}$ in* $\mathrm{Aut}(M)$ *fixes the type p,*

(ii) *$\bar{c}$ has finitely many conjugates under the automorphisms of M which leave p fixed.*

Proof. Let then e be an imaginary of T such that $e = \bar{a}/\epsilon$ where $\epsilon(\bar{x}, \bar{y})$ is a 0-definable equivalence relation, and $\bar{a}$ is in $\mathfrak{C}$. Let $\mathbf{p}$ a global generic type in $\epsilon(\bar{x}, \bar{a})$.

By taking a small but sufficiently saturated model M (ω-saturation will do), containing $\bar{a}$ and such that $\boldsymbol{p}$ does not fork over M, we obtain a real tuple $\bar{c}$ with properties (i) and (ii) related to $\mathrm{Aut}(\mathfrak{C})$ and $\boldsymbol{p}$.

But then we have $e \in \mathrm{dcl}^{eq}(\bar{c})$, for if $\sigma \in \mathrm{Aut}_{\bar{c}}(\mathfrak{C})$, then $\boldsymbol{p}^{\sigma} = \boldsymbol{p}$ and this implies that $\epsilon(\bar{x}, \bar{a}) \wedge \epsilon(\bar{x}, \bar{a}^{\sigma})$ must be consistent, thus σ fixes e.

On the other side, the group $\mathrm{Aut}_e(\mathfrak{C})$ *transitively* permutes the generic global types of the formula $\epsilon(\bar{x}, \bar{a})$. Since these are but in a finite number, if $\mathrm{Aut}(\mathfrak{C})_{\boldsymbol{p}}$ denotes the stabiliser of the type $\boldsymbol{p}$ under the action of $\mathrm{Aut}_e(\mathfrak{C})$, then the index of $\mathrm{Aut}(\mathfrak{C})_{\boldsymbol{p}}$ in $\mathrm{Aut}_e(\mathfrak{C})$ is finite. By the hypothesis, $\bar{c}$ has a finite orbit under $\mathrm{Aut}(\mathfrak{C})_{\boldsymbol{p}}$, then it has necessarily a finite orbit under $\mathrm{Aut}_e(\mathfrak{C})$. This gives $\bar{c} \in \mathrm{acl}^{eq}(e)$.

For the converse statement, (WEI) implies that for any type p over a model M, we can find a real tuple $\bar{c}$ such that

$$\begin{cases} \bar{c} \in \mathrm{acl}^{eq}(Cb(p)) \\ Cb(p) \in \mathrm{dcl}^{eq}(\bar{c}). \end{cases} \tag{1.6}$$

These properties imply (i) and (ii) above.

□

A real finite set with property (1.6) above will be found – for types of self-sufficient tuples – in Lemma 2.3.21 of Chapter 2.

The following result from [Pil95] will also be useful

Fact 1.3.10. *Assume $M \preccurlyeq \mathfrak{C}$ is a model of a stable theory, $\mathfrak{C}$ its monster model and let c, d be tuples in $\mathfrak{C}^{eq}$.*

If any of the following two conditions

(i) $c \in \mathrm{acl}(d)$

(ii) $c \mathop{\smile\!\!\!\!\!\!|}^{f}_{d} M$

holds, then $Cb(c/M) \subseteq \mathrm{acl}^{eq}(Cb(d/M))$.

We recall next the definition of CM-triviality.

Definition 1.3.11. A theory T is said to be *CM*-trivial, if for any algebraically closed sets $B \subseteq A$ of the monster $\mathbb{C}^{eq}$ of T, and all tuple c in $\mathbb{C}^{eq}$ with $\mathrm{acl}^{eq}(B, c) \cap A = B$ we always have $Cb(c/B) \subseteq \mathrm{acl}^{eq}\,(Cb(c/A))$.

With Pillay's [Pil95, Corollary 2.5], we can rephrase the definition above in terms of models of T and *real* tuples:

Fact 1.3.12. *A theory T is CM-trivial iff for all small models $M \preccurlyeq N$ and (real) tuples $\bar{c}$ from $\mathbb{C}$ with $acl(M,\bar{c}) \cap N = M$ we have $Cb(\bar{c}/M) \subseteq \operatorname{acl}^{eq}(Cb(\bar{c}/N))$.*
Moreover T is CM-trivial iff it is such after adding some set of parameters to T.

Such a property for T prevent the theory from interpreting fields:

Fact 1.3.13 ([Pil95, Proposition 3.2])**.** *No infinite field is interpretable in a CM-trivial theory T.*

1.4 Nilpotent Groups and graded Lie Algebras

We collect in this section some facts and notations from group theory and Lie algebras. We give a picture of the *Magnus-Lazard* correspondence between groups and Lie rings.

We refer to the (group) word $\gamma_k(x_1,\dots,x_k)$ as the *left-normed* or *simple* group commutator of length k

$$[x_1,\dots,x_k] = [[[\dots[[x_1,x_2],x_3],\dots],x_{k-1}],x_k] \tag{1.7}$$

where $\gamma_1(x) = x$ and $\gamma_2(x_1,x_2)$ is $[x_1,x_2] = x_1^{-1}x_1^{x_2} = x_1^{-1}x_2^{-1}x_1x_2$. We will not formally distinguish between group commutators and Lie brackets in the sequel.

For any group G, with $\gamma_k(G)$ we denote the verbal subgroup of G determined by γ_k, that is $\langle [g_1,\dots,g_k] \mid g_i \in G\rangle$. The subgroups $(\gamma_k(G))_{k<\omega}$ forms the so called *lower central series* of G, the most rapidly descending central series of G

Recall that in general a (descending) central series in the group G, is a chain $(H^i)_{1\leq i<\omega}$ of subgroups of G such that $H^i \supseteq H^{i+1}$, $H^1 = G$ and $[H^i, G] \subseteq H^{i+1}$.

We dually define the *upper central series* $(\zeta_k(G))_{k<\omega}$, where each subgroup $\zeta_k(G)$ can be defined as the set $\{g \mid [g,h_1,\dots,h_k] = 1, \forall h_i \in G\}$ for all $k < \omega$. For these, and related notions we refer to [Khu93, Rob96].

We denote by $\mathfrak{N}^c$ the variety defined by the word γ_{c+1}: the class of groups G with $\gamma_{c+1}(G) = 1$. These are by definition all groups of *nilpotency class* (at most) c. We have $\mathfrak{N}^c \subseteq \mathfrak{N}^{c+1}$ for all $c < \omega$, and we may call for short *nil-c* groups, the objects of $\mathfrak{N}^c$.

If p is a prime, by $\mathfrak{N}^c_p$ we denote the variety defined by the words $\gamma_{c+1}(\bar{x})$ and x^p, this is the class of all nilpotent groups of class c and of bounded exponent p.

The lower central series, is in particular a *Lazard series*, that is a decreasing chain $(H^n)_{n<\omega}$ of subgroups in G, with $H^1 = G$ and $[H^i, H^j] \leqslant H^{i+j}$ for all $i, j < \omega$. The properties we are going to state for the lower central series, also hold for Lazard series in general. Any Lazard series is a central series.

For any group G and all $k < \omega$, set for short G^k to be $\gamma_k(G)$ in the sequel. The series $G = G^1 \geqslant G^2 \geqslant \dots \geqslant G^k \geqslant \dots$ gives rise to a Lie ring associated to the group. This will be discussed below, following [Khu93, §3.2] or the first chapter of [Laz54].

Remark 1.4.1. Let x, y, z be elements of the group G. The following well known identities hold:

$$[x, y] = [y, x]^{-1} \tag{1.8}$$
$$[xy, z] = [x, z][x, z, y][y, z] \tag{1.9}$$
$$[x, y, z^x][y, z, x^y][z, x, y^z] = 1 \tag{Witt's identity}$$

Now for all $i < \omega$ consider the sections

$$\mathsf{gr}_i G := G^i/G^{i+1} \quad \text{and define} \quad \mathsf{gr}(G) := \bigoplus_{0 \le i < \omega} \mathsf{gr}_i G \tag{gr}$$

where $\mathsf{gr}_0 G := \mathbf{0}$.

The above remarks and (1.9) provide $\mathsf{gr}(G)$ with a natural non-associative ring structure $(\mathsf{gr}(G), +, [\ ,\ ], \mathbf{0})$ where the sum is the componentwise quotient group operation (written additively) and the product is induced by the group commutator:

$$[\boldsymbol{u}, \boldsymbol{v}] = \sum_k \left(\sum_{i+j=k} \overline{[u_i, v_j]} \right) \quad \text{for } \boldsymbol{u} = (\bar{u}_i) \text{ and } \boldsymbol{v} = (\bar{v}_i).$$

Now by (1.8) we have $[a, b] = -[b, a]$ for all a, b in $\mathsf{gr}(G)$ and by virtue of Witt's identity, the *Jacobi rule*

$$J(a, b, c) := [[a, b], c] + [[b, c], a] + [[c, a], b] = 0 \tag{1.10}$$

holds for all a, b, c. That is $\mathsf{gr}(G)$ is a *Lie ring* (a Lie $\mathbb{Z}$-algebra) according to the next definition. Notice that $\mathsf{gr}(G) \simeq \mathsf{gr}(G/\bigcap_{n<\omega} G^n)$.

Definition. If $\boldsymbol{k}$ is a commutative unitary ring, a *Lie algebra L over $\boldsymbol{k}$* or a Lie $\boldsymbol{k}$-algebra is a $\boldsymbol{k}$-module endowed with a k-bilinear map $[\ ,\ ]$ which factorises through $\bigwedge^2 L$, that is $[a, a] = 0$ for all a in L, and such that the Jacobi identity $J(a, b, c) = \mathbf{0}$ is satisfied for all a, b, c in L.

For a subset S of a Lie $\boldsymbol{k}$-algebra L we denote by $\langle S \rangle$ or $\langle S \rangle^L$ the subalgebra generated by S in L while $\langle S \rangle_{\boldsymbol{k}}$ denotes the $\boldsymbol{k}$-submodule of L generated by S. The product $[S, T]$ of subsets S and T of L is $\langle [s, t] \mid s \in S, t \in T \rangle_{\boldsymbol{k}}$, while the ideal generated in L by S is denoted by $\langle S \rangle_{\mathsf{id}}$. This is – by means of anti-commutativity and repeated applications of the Jacobi identity – also $\langle S, [S, L], [S, L, L], \ldots \rangle_{\boldsymbol{k}}$.

Exactly like for groups, we define the terms of the lower central series $\gamma_n(L)$ of L recursively as $[\gamma_{n-1}(L), L]$ for all $n < \omega$ where $\gamma_1(L) = L$. These builds a decreasing chain of ideals of L and $\gamma_n(L) = \langle \gamma_n(s_1, \ldots, s_n) \mid s_i \in L \rangle_{\boldsymbol{k}}$. The definition for the upper central terms $\zeta_n(L)$ is exactly the same defined above.

We say that L is *nilpotent of class* (at most) c, if $\gamma_{c+1}(L) = \mathbf{0}$.

If a Lie $\boldsymbol{k}$-algebra L is generated by a set S, then an inductive argument on Jacobi identities shows, that L is generated as a $\boldsymbol{k}$-module, by all the *simple monomials* with entries in S, that is by left-normed products $[s_1, \ldots, s_k]$ like (1.7) of *weight* k, for all $k < \omega$. In particular $\gamma_n(L)$ is the ideal generated by all simple monomials of length n in elements of S or the $\boldsymbol{k}$-module generated by monomials of weight $\geq n$.

Remark 1.4.2. For any group G, $\mathsf{gr}(G)$ is generated as a ring, by the *abelianised* group G_{ab}: the quotient of G modulo $G' = \gamma_2(G)$. This follows by the natural surjective map of $\otimes^n_{\mathbb{Z}} G_{ab}$ onto $\mathsf{gr}_n G$ (see [Khu93, §2]) induced by the group commutator.

In particular $\mathsf{gr}(G)$ is a *graded algebra*: it is the direct sum of its homogeneous submodules $\mathsf{gr}_i G$ *of weight* i *in* G_{ab} and $[\mathsf{gr}_i G, \mathsf{gr}_k G] \subseteq \mathsf{gr}_{i+k} G$ for all i, k.

If G is a group in $\mathfrak{N}^c_p$, the Lie ring $\mathsf{gr}(G)$ carries a Lie $\mathbb{Z}_p$-algebra structure which is nilpotent of class c.

Free Algebras and basic commutators

For the following definitions we follow [Ser06] and [Bah78].

For any set X, the *free magma* $(\mathcal{M}(X), \cdot\,)$ is – roughly speaking – the image of X under the free functor $\mathcal{M}$ from *sets* to the category of all structures which interpret a binary operation.

The elements of $\mathcal{M}(X)$ which are referred to as *non-associative words over* X are the disjoint union of ω subsets $\mathcal{M}_n(X)$, each one collecting the *words of weight* or *length* n, for $n \geq 1$ (cfr. [Bou06, §2]). We have $\mathcal{M}_1(X) = X$ and the product $\cdot$ maps $\mathcal{M}_i(X) \times \mathcal{M}_k(X)$ into $\mathcal{M}_{i+k}(X)$.

Define the *free* $\boldsymbol{k}$*-algebra on* X as the free $\boldsymbol{k}$-module $\mathcal{F} = \mathcal{F}(X, \boldsymbol{k})$ with basis $\mathcal{M}(X)$ and with a $\boldsymbol{k}$-bilinear multiplication $\cdot$ extended from the product on $\mathcal{M}(X)$, which in particular makes $(\mathcal{F}, \cdot, \mathbf{0})$ a non-associative ring without unit, such that $(ta)\cdot b = t(a\cdot b) = a \cdot tb$, for all $t \in \boldsymbol{k}$ and all a, b from $\mathcal{F}$.

$\mathcal{M}(X)$ induces a natural grading on $\mathcal{F}$ given by $\mathcal{F} = \bigoplus_{n<\omega} \mathcal{F}_n$ for $\mathcal{F}_n := \langle \mathcal{M}_n(X) \rangle_{\boldsymbol{k}}$. Each element a of $\mathcal{F}$ is henceforth expressible in a unique manner as a finite sum $\sum a_n$, where $a_n \in \mathcal{F}_n$ are called the *homogeneous components* of a.

Let now $\mathcal{A}$ and $\mathcal{B}$ be the ideals of $\mathcal{F}$ respectively generated by the sets $\{(u \cdot v) \cdot w - u \cdot (v \cdot w) \mid u, v, w \in \mathcal{F}\}$ and $\{u \cdot u, J(u, v, w) \mid u, v, w \in \mathcal{F}\}$ where J is the homogeneous term associated to the Jacobi identity (1.10). We define

$$A^+(X, \boldsymbol{k}) = \mathcal{F}(X, \boldsymbol{k})/\mathcal{A} \quad \text{and} \quad L(X, \boldsymbol{k}) = \mathcal{F}(X, \boldsymbol{k})/\mathcal{B}$$

as respectively the *free associative* and the *free Lie* algebra *on* X *over* $\boldsymbol{k}$.

In [Bah78, 2.1] is proved, that both $\mathcal{A}$ and $\mathcal{B}$ are *homogeneous ideals*, that means $\mathcal{A} = \sum_i \mathcal{A} \cap \mathcal{F}_i$ and $\mathcal{B} = \sum_i \mathcal{B} \cap \mathcal{F}_i$.

It follows $A^+(X, \boldsymbol{k})$ and $L(X, \boldsymbol{k})$ inherit from $\mathcal{F}$ the grading:

$$A^+(X, \boldsymbol{k}) = \bigoplus_{i \geq 1} \mathcal{F}_i / \mathcal{A} \cap \mathcal{F}_i \quad \text{and} \quad L(X, \boldsymbol{k}) = \bigoplus_{i \geq 1} \mathcal{F}_i / \mathcal{A} \cap \mathcal{F}_i. \tag{1.11}$$

We denote by $A_i = A_i(X, \boldsymbol{k})$ and $L_i = L_i(X, \boldsymbol{k})$ the $\boldsymbol{k}$-submodules in the grading above, and call them *homogeneous submodules* of weight i.

A_i and L_i are generated by monomials of weight i: the images in A^+ and L of words in $\mathcal{M}_i(X)$. It is customary to speak of *degree* i for the elements of $A_i(X, \boldsymbol{k})$ instead of weight.

Remark 1.4.3. Any Lie $\boldsymbol{k}$-algebra M is the image of a free Lie algebra $L(X, \boldsymbol{k})$ for some X. Moreover any associative algebra A is endowed with a product $[a, b] = ab - ba$ which turns $(A, +, [\ ,\ ])$ into a Lie algebra. As a consequence, there exists a natural Lie $\boldsymbol{k}$-algebra homomorphism ϵ of $L(X, \boldsymbol{k})$ onto the Lie subalgebra of $A(X, \boldsymbol{k})$ generated by X such that $\epsilon\colon x \to x$ for all $x \in X$.

Whether for the $\boldsymbol{k}$-module $A(X, \boldsymbol{k})$ a $\boldsymbol{k}$-basis is provided by all ordered products over X, for $L(X, \boldsymbol{k})$ we recur to the so called basic monomials, also called *Hall's Families*. In the groups context the very same definition applies to *basic commutators*.

Definition 1.4.4 (Basic Monomials)**.** We inductively construct a linearly ordered set of Lie monomials $\mathscr{B} = \bigcup_{n \geq 1} \mathscr{B}_n$, where each $\mathscr{B}_n \subseteq L_n(X, \boldsymbol{k})$ will be called the set of *basic* monomials of weight n.

Let $\mathscr{B}_1$ coincide with some linear order on X.

Assume a set of *basic monomials* $\mathscr{B}_{<n} = \bigcup_{i<n} \mathscr{B}_k$ *of weight less than* n has been defined and totally ordered, by choosing a linear ordering for each $\mathscr{B}_k$ and following the rule: $a > b$ holds whenever the weight of a is greater than the weight of b.

Now consider any pair of monomials u, v in $\mathscr{B}_{<n}$, the sum of whose weights is n. Then the product $[u, v]$ is a *basic monomials of weight* n and lays in $\mathscr{B}_n$ if both of the following conditions are satisfied:

- $u > v$,
- if $u = [z, w]$ for $z, w \in \mathscr{B}_{<n}$, then $w \leq v$.

The following result is referred to as *Hall's Basis Theorem.*[3]

Theorem 1.4.5 ([Hal50],[Bah78, Theorem 2.2.1])**.** *Let $\mathscr{B}$ a set of basis commutators on X in $L = L(X, \boldsymbol{k})$. Then L is a free $\boldsymbol{k}$-module with basis $\mathscr{B}$.*

In particular each homogeneous submodule $L_n(X, \boldsymbol{k})$ is free, with basis $\mathscr{B}_n$ for all $n \geq 1$.

For a clear account of the group theoretical analogous around Hall's Theorem and the *collecting process* we refer to [Khu93].

As a corollary to the above theorem in [Bah78] we find

Fact 1.4.6. *The canonical Lie morphism ϵ of Remark 1.4.3 mapping $L(X, \boldsymbol{k})$ into $A(X, \boldsymbol{k})$ is a Lie algebra embedding.*

[3]Although ambiguous, the name doesn't harm the fatherhood of both Philip and Marshall.

To the former, one attributes the so called *collecting process* (see [Hal59]), from which Definition 1.4.4 arise. It is an algorithm to stepwise transform a group word into an ordered expression of basic commutators. In [Hal50], Marshall Hall describes a collecting process in the context of Lie rings but claims the same results to hold for Lie algebras over any field.

The proof in [Bah78] – attributed to A.I. Shirshov – holds for arbitrary commutative rings.

The above ϵ, coincide with the canonical map of a Lie algebra L in its *universal envelope* $U(L)$ (see also [Ser06]), this is always an embedding provided L is a free $\boldsymbol{k}$-module.

The following fact will also be needed.

Fact 1.4.7 ([Bah78, Lemma 2.3.3])**.** *Let L be the free Lie algebra $L(X, \boldsymbol{k})$ over the set X. If $\mathcal{B}$ is a basis for the free $\boldsymbol{k}$-submodule $\langle X\rangle_{\boldsymbol{k}} = L_1(X, \boldsymbol{k}) = \mathcal{F}_1(X, \boldsymbol{k})$ of L, then $L(\mathcal{B}, \boldsymbol{k}) = L$.*

We introduce below the class of nilpotent Lie algebras, which are to be associated a notion of Hrushowski predimension in Chapters 2 and 3. We show that these structures isolate exactly those algebras which arise from $\mathfrak{N}_p^c$-groups as images under the functor gr.

Definition 1.4.8. For a prime number p and a positive integer c, we define by $\mathfrak{L}_p^c$ the class of all c-nilpotent (graded) Lie algebras M over the p-element field $\mathbb{Z}_p$, which satisfy the following properties:

1. there are $\mathbb{Z}_p$-subspaces M_i for $1 \le i \le c$ with $M = M_1 \oplus \cdots \oplus M_c$,
2. $[M_i, M_j] \subseteq M_{i+j}$ for all i, j (M_k is defined to be $\mathbf{0}$ for $k > c$),
3. $M = \langle M_1 \rangle$

Remark 1.4.9. The whole *grading* $(M_i)_{i<\omega}$ depends indeed only on the *choice* for the space M_1: by property 3. above each subspace M_i is the $\mathbb{Z}_p$-subspace of M generated by simple monomials of weight i in the elements from (a $\mathbb{Z}_p$-basis of) M_1.

Lie subalgebras of an $\mathfrak{L}_p^c$-algebra M are not always again $\mathfrak{L}_p^c$-objects, we hence *define* an $\mathfrak{L}_p^c$-subalgebra H of M, if $H = \langle H_1 \rangle^M$ for some $\mathbb{Z}_p$-subspace H_1 of M_1. By an $\mathfrak{L}_p^c$-morphisms we mean a graded homomorphism of Lie algebras. That is, if $\phi\colon M \to N$ for $M, N \in \mathfrak{L}_p^c$, then $\phi(M_i) \subseteq N_i$ for all i.

Remark. As observed above, we get a correspondence

$$\mathsf{gr}\colon \mathfrak{N}_p^c \to \mathfrak{L}_p^c \tag{1.12}$$

where, if $G \in \mathfrak{N}_p^c$ and M denotes $\mathsf{gr}(G)$, G_{ab} corresponds to M_1 of Definition 1.4.8. Since the terms of the lower central series are fully invariant, the map above is a functor provided we allow Lie morphisms in general. Group homomorphisms may have non-graded images under gr.

In the next section, it will be shown however that gr is onto of the respective objects.

For a first-order treatment of $\mathfrak{L}_p^c$ we choose the signature $\mathscr{L}^c$ consisting of ring symbols $\mathbf{0}$, $+$ and $[\ ,\ ]$, of the scalar functions from $\mathbb{Z}_p$ and of predicates P_i which are interpreted by the grading ($P_i(M) = M_i$). Notice that $\mathfrak{L}_p^c$ is not an elementary $\mathscr{L}^c$-class. Property 3. (as opposed to 1. and 2.) of Definition 1.4.8 cannot be expressed at the first order in $\mathscr{L}^c$ unless a bound to the length of sums in M is given.

By the previous discussion, for any set X, the *free c-nilpotent Lie $\mathbb{Z}_p$-algebra over X*, which we define as

$$L^c(X) := L(X,\mathbb{Z}_p)/\gamma_{c+1}(L(X,\mathbb{Z}_p))$$

is an object of $\mathfrak{L}_p^c$. This is for $L(X,\mathbb{Z}_p)$ is a graded Lie algebra and $\gamma_{c+1}(L(X,\mathbb{Z}_p))$ is an homogeneous ideal which is equal to $\sum_{i>c} L_i(X,\mathbb{Z}_p)$.

Similarly, for any object M of $\mathfrak{L}_p^c$, $\gamma_i(M) = \sum_{j\geq i} M_j$.

As a corollary to Hall's Theorem above we get (cfr. [Khu93, Corollary 2.7.3])

Fact 1.4.10. *For any given set $\mathscr{B}$ of basic monomials over X, denote by $\mathscr{B}_{\leq c}$ the set of elements in $\mathscr{B}$ of weight not greater than c. Then $\mathscr{B}_{\leq c}$ is a $\mathbb{Z}_p$-basis of $L^c(X)$ and in particular every element w of $L^c(X)$ admits a unique linear combination*

$$w = \sum_{b\in\mathscr{B}_{\leq c}} s_b b \quad \text{with } b \text{ in } \mathscr{B}_{\leq c} \text{ and non-trivial } s_b \in \mathbb{Z}_p \tag{BC}$$

Definition 1.4.11. If $\mathscr{B}_{\leq c}$ and $L^c(X)$ are as above, we define the *support* of an element $w \in L^c(X)$, as the minimal subset $\operatorname{supp}(w)$ of X, for which each basic monomial b in the sum (BC) above, carries entries from $\operatorname{supp}(w)$ according to Definition 1.4.4. We may specify the set over which $\mathscr{B}$ is constructed, by writing $\operatorname{supp}_X(w)$.

If a basic monomial $b \in \mathscr{B}$, has support in a subset Y of X, we refer to b also as a monomial *over* Y, or shortly, as a basic Y-monomial.

Remark 1.4.12. With Fact 1.4.7 and Remark 1.4.3, $L^c(\,\cdot\,)$ may be seen as a free functor of $\mathbb{Z}_p$-vector spaces into $\mathfrak{L}_p^c$-algebras, adjoint to the predicate P_1 of $\mathscr{L}^c$: for any $\mathbb{Z}_p$-vector space V and $\mathfrak{L}_p^c$-algebras M, we have – with the obvious maps – a bijection

$$\operatorname{Hom}_{\mathbb{Z}_p}(V, P_1(M)) \to \operatorname{Hom}_{\mathfrak{L}_p^c}(L^c(V), M). \tag{1.13}$$

In particular any object M of $\mathfrak{L}_p^c$ is the quotient of $L^c(M_1)$ modulo an homogeneous ideal. In the above notations $R = \sum_{i\leq c} L_i(M_1,\mathbb{Z}_p)\cap R$ with $M_1 \cap R = \mathbf{0}$. We will write $R = R_2 + \cdots + R_c$.

Since the subspace M_1 is intrinsic to the structure M, the choice of the relators ideal R may be regarded as canonically associated to M.

By a *$\mathfrak{L}_p^c$-presentation* of M we denote both the expression $M = \langle M_1 \mid R\rangle$ and the associated exact sequence

$$R \longmapsto L^c(M_1) \overset{\epsilon_M}{\longmapsto} M \tag{1.14}$$

On the other hand, to any homogeneous ideal R of $L = L^c(X)$, the quotient L/R is an object of $\mathfrak{L}_p^c$.

We say that M is finitely generated if M_1 has finite $\mathbb{Z}_p$-dimension, hence exactly if M_1 (and M) is finite. Note that in the category $\mathfrak{L}_p^c$ the notion of *finitely presented* (that is M_1 and R are finite dimensional) coincide with being finitely generated. The same holds in general for nilpotent groups.

As a result of Definition 1.4.8, morphisms among objects of $\mathfrak{L}_p^c$ aren't richer than those among their generating $\mathbb{Z}_p$-vector spaces.

Lemma 1.4.13. *For any M and N in $\mathfrak{L}_p^c$, to any $\mathfrak{L}_p^c$-morphism ϕ of M to N, there is a unique $\widehat{\phi} \in \mathrm{Hom}_{\mathfrak{L}_p^c}(L^c(M_1), L^c(N_1))$ which makes the square below commute.*

$$\begin{tikzcd} L^c(M_1) \arrow[r, "\widehat{\phi}"] \arrow[d, "\epsilon_M"] & L^c(N_1) \arrow[d, "\epsilon_N"] \\ M \arrow[r, "\phi"] & N \end{tikzcd} \tag{1.15}$$

Relations between the $\mathfrak{N}_p^c$-free group and the free $\mathfrak{L}_p^c$-algebra

Before we prove that (1.12) is onto, we first establish a correspondence between the free objects in the classes $\mathfrak{L}_p^c$ and $\mathfrak{N}_p^c$.

Let $A^+(X)$ be the free associative algebra $A^+(X, \mathbb{Z}_p)$ over $\mathbb{Z}_p$ defined above. We add a multiplicative unit – and hence *elements of zero degree* – by defining $A(X)$ to be $\mathbb{Z}_p \oplus A^+(X)$ and extending addition and multiplication in the natural way. $A(X)$ inherits the grading (1.11) and we set $A_0(X) = \mathbb{Z}_p$.

Let $A^c(X)$ be the quotient algebra of $A(X)$ modulo the ideal $\sum_{i>c} A_i(X)$, that is the *free* unitary associative nilpotent algebra of class c. In particular

$$A^c(X) \simeq A_0(X) \oplus A_1(X) \oplus \cdots \oplus A_c(X).$$

Let now $F_p(X)$ denote the free group of exponent p on the set X, then $F_p^c(X) := F_p(X)/\gamma_{c+1}(F_p(X))$ is the free group in $\mathfrak{N}_p^c$.

Now assume $c < p$, since $(1+x)^p = 1+x^p = 1$ in $A^c(X)$, one can extend the map $X \ni x \mapsto 1+x$ to a group homomorphism ϕ of $F_p(X)$ onto the subgroup $\langle 1+x \mid x \in X\rangle$ of the units of $A^c(X)$ (the multiplicative inverse of $1+a$ being $1-a+a^2-a^3+\cdots+(-1)^{c-1}a^{c-1}$).

If we put together [Wit36],[Mag37],[Mag40],[Hal59, Lemma 11.2.2] and [Ser06, Theorem 6.3], we find that the nucleus of ϕ coincides with $\gamma_{c+1}(F_p(X))$ and the following facts hold.

Fact 1.4.14. *If we assume $c < p$ we have an injective group homomorphism ϕ of $F_p^c(X)$ into the units of $A^c(X)$ extending $x \to 1+x$ for $x \in X$.*

For all words w of $F_p^c(X)$ we set

$$\phi\colon w \longmapsto 1 + \lambda(w) + W$$

where $\lambda(w)$ is the homogeneous component $\phi(w)_n \in A_n(X)$ of $\phi(w)$ of minimal positive degree $n \leq c$ such that $\phi(w)_1 = \cdots = \phi(w)_{n-1} = \mathbf{0}$ and W is a sum of components of *higher degree. We also say that n is the* weight *of w and write* $\mathsf{w}(w) = n$. *$\lambda(w)$ is called the* leading term *of $\phi(w)$ and has the properties:*

- *$\lambda(gh)$ is $\lambda(g)$ or $\lambda(h)$ according to which among g and h has lower weight. If g and h have the same weight and $\lambda(g) + \lambda(h) \neq \mathbf{0}$, then $\lambda(gh) = \lambda(g) + \lambda(h)$.*

- *$\lambda(g^{-1}) = -\lambda(g)$.*

- *If $[\lambda(g), \lambda(h)] \neq \mathbf{0}$, then $\lambda([g,h]) = [\lambda(g), \lambda(h)]$. $\mathsf{w}([g,h]) \geq \mathsf{w}(g) + \mathsf{w}(h)$ and if $[\lambda(g), \lambda(h)] = \mathbf{0}$, then $\mathsf{w}(g) = \mathsf{w}(h)$ and $\lambda(g^h) = \lambda(g)$.*

- *$\gamma_i(F_p^c(X))$ coincides with the set of all g in $F_p^c(X)$ with $\mathsf{w}(g) \geq i$.*

Here above, $[\lambda(g), \lambda(h)]$ denotes the Lie product $\lambda(g)\lambda(h) - \lambda(h)\lambda(g)$ in the associative algebra $A^c(X)$.

If ϵ is the canonical embedding of Fact 1.4.6, since we have $\epsilon(\gamma_k(L(X))) = \epsilon(L(X)) \cap \sum_{i \geq k} A_i(X)$ for all $1 \leq k$, ϵ factorises to a Lie monomorphism of the free nilpotent Lie algebra $L^c(X)$ into $A^c(X)$, hence we identify $L^c(X)$ with the Lie subalgebra generated by X in $A^c(X)$.

By the above facts we obtain a map

$$\lambda\colon F_p^c(X) \to L^c(X) \tag{1.16}$$

which gives rise to a well defined *injective* $\mathfrak{L}_p^c$-morphism

$$\begin{aligned} \bar{\lambda}\colon \mathsf{gr}(F_p^c(X)) &\longrightarrow L^c(X) \\ \sum \bar{u}_i &\longmapsto \sum \lambda(u_i) \end{aligned}$$

which maps X identically onto X.

Since on the contrary, $\mathsf{gr}(F_p^c(X))$ is the image of an epimorphism of $L^c(X)$ which extends the identity on X, it follows

Remark 1.4.15. $\mathsf{gr}(F_p^c(X))$ and $L^c(X)$ are $\mathfrak{L}_p^c$-isomorphic via $\bar{\lambda}$.

Notice that the condition $p > c$ is necessary. We have for instance, in $F_p(X)$, that Engel elements $[x, y, \ldots, y]$ of length p are congruent to 1 modulo $\gamma_{p+1}(F_p(X))$ for all x, y (cfr. [Khu93, Theorem 2.8.11]).

Retrieving groups from $\mathfrak{L}_p^c$-algebras

Let M be a Lie algebra of $\mathfrak{L}_p^c$ with $p > c$. As observed above M is isomorphic to the quotient $L^c(X)/R$, where X is a $\mathbb{Z}_p$-basis of M_1 and R is a homogeneous ideal of $L^c(X)$.

The idea is to associate M to a quotient of $F_p^c(X)$: we need to find a suitable normal subgroup. Consider the map (1.16) above and define

$$N = \left\{ w \in F_p^c(X) \mid \lambda(w) \in R \right\}$$

then by Fact 1.16, as $\lambda(hg^{-1})$ equals $\lambda(h)$ or $-\lambda(g)$ or again $\lambda(h)-\lambda(g)$, N is a subgroup of $F_p^c(X)$.

Moreover, the same fact implies that for all g in N and all x in X, either $\lambda(g^x) = \lambda(g)$ or $\lambda([g,x]) = [\lambda(g), x]$ is in the ideal R. This yields that N is a normal subgroup of $F_p^c(X)$. Hence the quotient $F_p^c(X)/N$ which we denote by $\mathscr{G}(M)$, is a group in the variety $\mathfrak{N}_p^c$.

We can now prove the following somewhat dual result to [Mag40, I.].

Proposition 1.4.16. *Let p be a prime number greater than c. With the above definition and Lemma 1.4.13, the map $M \mapsto \mathscr{G}(M)$ is a functor of $\mathfrak{L}_p^c$ into $\mathfrak{N}_p^c$-groups such that $\mathsf{gr}(\mathscr{G}(M)) \simeq_{\mathfrak{L}_p^c} M$ for all M in $\mathfrak{L}_p^c$.*

For a fixed M in $\mathfrak{L}_p^c$, then $\mathfrak{L}_p^c$-subalgebras (ideals) of M correspond – via λ – to subgroups (normal subgroups) of $\mathscr{G}(M)$.

Proof. Assume $M = \langle M_1 \mid R\rangle$ and X is a $\mathbb{Z}_p$-basis of M_1. Put $F = F_p^c(X)$ and let $G = \mathscr{G}(M)$ be the quotient of F modulo the subgroup N defined above.

Since R is a homogeneous ideal and $R = R_2 + \cdots + R_c$, then $M_n \simeq_{\mathbb{Z}_p} L_n(X)/R_n$, for all $n \leq c$ and if F^n denotes $\gamma_n(F)$ then, as abelian groups

$$\mathsf{gr}_n G = \gamma_n(G)/\gamma_{n+1}(G) \simeq F^n/F^{n+1}(F^n \cap N) \simeq \frac{\mathsf{gr}_n F}{F^{n+1}(F^n \cap N)/F^{n+1}}.$$

On the other hand, Remark 1.4.15 and the definition of N imply that the $\mathbb{Z}_p$-isomorphism $\bar{\lambda}\colon F^n/F^{n+1} \to L_n(X)$ maps $F^{n+1}(F^n \cap N)/F^{n+1}$ exactly onto R_n. It follows M_n is isomorphic to $\gamma_n(G)/\gamma_{n+1}(G)$ as a $\mathbb{Z}_p$-vector space, for all n. Moreover, since $\sum_n F^{n+1}(F^n \cap N)/F^{n+1}$ is an ideal of $\mathsf{gr}(F) \simeq L^c(X)$, then $\mathsf{gr}(G)$ is $\mathfrak{L}_p^c$-isomorphic to M.

The remaining statements directly descend from the construction of $\mathscr{G}(M)$.

□

The Baker-Hausdorff Formula

There is a second and more classical way to reconstruct groups from Lie algebras, which has a topological-analytical approach. In our nilpotent context, this yields a more effective model-theoretical interpretation of the aforementioned correspondence. To describe this method, we have to restart from the original Witt's *Treue Darstellung* [Wit36] in characteristic zero.

We mention here that a filtration $(\mathfrak{g}_i)_{i<\omega}$ of a $\boldsymbol{k}$-algebra $\mathfrak{g}$ is a decreasing series of ideals $\mathfrak{g}_i$ with $\mathfrak{g}_i \cdot \mathfrak{g}_j \subseteq \mathfrak{g}_{i+j}$.

In fact the lower central series $(\gamma_k(L))_k$ of a Lie algebra L constitutes an example of (central) filtration. We say that $\mathfrak{g}$ is *separated* with respect to the filtration $(\mathfrak{g}_i)$ if $\bigcap \mathfrak{g}_i = \mathbf{0}$. A separating filtration induces an Hausdorff (T_2) topology on $\mathfrak{g}$. We refer to [Bou06, Laz54] for these notions.

Consider the Magnuss algebra ([Bou06, §5.1]) $\widehat{A} = \widehat{A}(X, \mathbb{Q})$ over the rationals. This is the topological completion of the free associative unitary $\mathbb{Q}$-algebra

$$A = A(X, \mathbb{Q}) = \mathbb{Q} \cdot 1 \oplus A^{+}(X, \mathbb{Q})$$

with respect to the topology induced by the natural degree-filtration.

Elements of $\widehat{A}$ are noncommutative formal power series in the *indeterminates* X and coefficients in $\mathbb{Q}$:

$$a = \sum_{i<\omega} a_i \qquad \text{for } a_i \in A_i(X, \mathbb{Q}),\ a_0 \in \mathbb{Q}.$$

As $L(X, \mathbb{Q})$ is identified with the Lie subalgebra of $A(X, \mathbb{Q})$ generated by X, we define the elements of $\widehat{L}$ as the formal series $\sum_{i\geq 1} b_i$ of $\widehat{A}$ with each homogeneous component b_i belonging to $L_i(X, \mathbb{Q})$.

If $\mathfrak{m}$ denotes the ideal $\sum_{i\geq 1} A_i(X)$ of $\widehat{A}$, then $1 + \mathfrak{m}$ is a multiplicative group. We obtain the continuos bijections (cfr.[Ser06, IV.7])

$$\begin{aligned} \exp\colon \mathfrak{m} &\longrightarrow 1 + \mathfrak{m} & \qquad \log\colon 1 + \mathfrak{m} &\longrightarrow \mathfrak{m} \\ a &\longmapsto \sum_{i\geq 0} \frac{a^i}{i!} & 1 + b &\longmapsto \sum_{i\geq 1} (-1)^{i+1} \frac{b^i}{i} \end{aligned}$$

with the usual properties $\log(\exp a) = a$ and $\exp(\log(1 + b)) = 1 + b$.

Fact 1.4.17 ([Ser06, 1.IV.7][Bah78, Theorem 6.1,1])**.** *In the above notations,* $\exp(\widehat{L})$ *is a multiplicative subgroup of* $1 + \mathfrak{m}$*.*

Moreover if ϵ *denotes the homomorphism of the free group* $F(X)$ *on* X *into* $1 + \mathfrak{m}$ *which extends* $x \mapsto \exp(x)$ *for all* x *in* X*, then* $\epsilon \log$ *is a goup monomorphism of* $F(X)$ *into* $(\widehat{L}, \circ)$ *where* $\circ$ *is the group law on* $\widehat{L}$ *given by*

$$\xi \circ \eta = \log(\exp(\xi) \exp(\eta))$$

for all $\xi, \eta \in \widehat{L}$*.*

Theorem 1.4.18 ([Bah78],[Bou06, Proposition §5.4],[Laz54, Théorème 4.2])**.** *Let now* X *be the set* $\{x, y\}$ *the element of* $\widehat{A}(x, y, \mathbb{Q})$*, then in the previous notation, we have*

$$x \circ y =: \mathsf{H}(x, y) = \sum_{i=1}^{\infty} t_i \mathsf{h}_i(x, y) \tag{1.17}$$

where $\mathsf{h}_i(x, y)$ *is a homogeneous term in* $L_i(\{x, y\}, \mathbb{Z})$ *of total weight* i *in* x *and* y *and* $t_i \in \mathbb{Q}$*.*

For any complete, separated, filtered Lie Algebra $\mathfrak{g}$ *with filtration* $(\mathfrak{g}_\alpha)$*, over a characteristic zero field* $\boldsymbol{k}$*, the map*

$$\begin{aligned} \circ\colon \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (a, b) &\longmapsto \mathsf{H}(a, b) \end{aligned} \tag{1.18}$$

induces a group structure *on* $\mathfrak{g}$ *compatible with the topology and such that*

- *the neutral element of* $(\mathfrak{g}, \circ)$ *is the additive zero* $\mathbf{0}$ *and for any element* m*, the* $\circ$*-inverse* m^{-1} *of* m *coincides with the additive inverse* $-m$*. Moreover the* n*-th power* a^n *in* $\circ$ *of any element* a *of* $\mathfrak{g}$ *is* $n \cdot a$ *for all* $n \in \mathbb{Z}$

- *the* group commutator $[l, m]$ *built from the group operation* $\circ$ *equals the Lie product* $[l, m]$ *in* $\mathfrak{g}$ *modulo the ideal* $\mathfrak{g}_{\alpha+1}$ *provided* l *or* m *is in* $\mathfrak{g}_\alpha$*.*

- *the chain* $(\mathfrak{g}_\alpha)$ *becomes a central series of* $(\mathfrak{g}, \circ)$*. The quotient group operation induced by* $\circ$ *on* $\mathfrak{g}_\alpha/\mathfrak{g}_{\alpha+1}$ *coincides with the abelian structure of the quotient algebras.*

For an explicit calculation of the terms $s_i \mathsf{h}_i(x, y)$ in (1.17) one may see [Ser06, IV.8]. A first segment of $\xi \circ \eta$ is given by

$$\xi \circ \eta = \xi + \eta + \frac{1}{2}[\xi, \eta] - \frac{1}{12}([\xi, \eta, \eta] + [\eta, \xi, \xi]) + \cdots \tag{1.19}$$

Now the crucial fact which allows us to apply the above machinery to $\mathbb{Z}_p$-algebras in $\mathfrak{L}_p^c$ for $p > c$, is the following observation.

Fact 1.4.19 ([Mag40, Laz54])**.** *Let* $\mathbb{Q}_c$ *denote the subring of* $\mathbb{Q}$ *which consists of all quotients* r/s *for coprime* r, s *such that if a prime* q *divides* s*, then* $q \leq c$*. In (1.18) above we have* $t_i \in \mathbb{Q}_i$ *for all* $i < \omega$*.*

Notice that, since $p > c$ as a $\mathbb{Z}_p$-vector space any object M in $\mathfrak{L}_p^c$ carries a $\mathbb{Q}_c$-algebra structure, simply letting $r/s \cdot m = \bar{r}\bar{s}^{-1}m$ where $\bar{r}$ and $\bar{s}$ denote r and s modulo p.

As observed in [Bou06], to a *finite* central filtration automatically corresponds a complete and separated (discrete) topology. This is the case for nilpotent algebras. In particular Theorem 1.4.18 and Fact 1.4.19 yield (cfr. [Laz54, Theorem II,4.2]):

Corollary 1.4.20. *For* $p > c$*, considering the lower central filtration on* $\mathfrak{L}_p^c$*-algebras, we obtain*

$$\begin{aligned} G \colon \mathfrak{L}_p^c &\longrightarrow \mathfrak{N}_p^c \\ M &\longmapsto G(M) = (M, \circ, \mathbf{0}) \end{aligned} \tag{1.20}$$

By Theorem 1.4.18 and by the definition of $\mathfrak{L}_p^c$*, for each such algebra* M*, since* $M = \langle M_1 \rangle$ *we have* $\mathsf{gr}(G(M)) = M$*. Moreover for any* $\mathfrak{L}_p^c$*-extension* $M \supseteq N$*, the corresponding groups* $G = G(M)$ *and* $H = G(N)$ *satisfy* $\gamma_k(H) = \gamma_k(G) \cap H$*. In particular* $\mathsf{gr}(H)$ *is an* $\mathfrak{L}_p^c$*-subalgebra of* $\mathsf{gr}(G)$*.*

Corollary 1.4.21. *For any* $\mathfrak{L}_p^c$*-algebra* M*, the group* $G(M)$ *is* definably interpretable *([Mar02, p.24]) in the* $\mathscr{L}^c$*-structure* M*.*

Remark 1.4.22. In Lemma 3.1 of [Bau96], a different approach is described, to reconstruct a group law from a class of $\mathbb{Z}_p$-vector spaces, identifiable with our $\mathfrak{L}_p^2$. That is motivated by the following instance of the collecting process, peculiar of nilpotency class 2.

Let G be a $\mathfrak{N}_p^2$-group and assume a subset $\{a_\alpha \mid \alpha < \kappa\} \subseteq G$ has been chosen, which – modulo G' – is a base of the $\mathbb{Z}_p$-vector space G_{ab}. Then, any element g of G writes in a unique way as a product $g = \prod_\alpha a_\alpha^{r_\alpha} x$ for some $x \in G'$ and (with the due precautions) $r_\alpha \in \mathbb{Z}_p$.

As $G' \subseteq Z(G)$, if $h = \prod_\alpha a_\alpha^{s_\alpha} y$, then $gh = \prod_\alpha a_\alpha^{r_\alpha + s_\alpha} \prod_{\alpha > \beta} [a_\alpha, a_\beta]^{r_\alpha s_\beta} xy$.

This yields a group operation $\bullet$ on $\mathsf{gr}(G)$ defined as follows:[4]

$$\Big(\sum\nolimits_\alpha r_\alpha \bar{a}_\alpha + x\Big) \bullet \Big(\sum\nolimits_\alpha r_\alpha \bar{a}_\alpha + y\Big) = \sum\nolimits_\alpha (r_\alpha + s_\alpha)\bar{a}_\alpha + x + y + \sum\nolimits_{\alpha > \beta} r_\alpha s_\beta [a_\alpha, a_\beta]$$

The peculiarity of this setting, is now the fact that $(\mathsf{gr}(G), \bullet)$ is now isomorphic– as a group – to G. If we define $\bullet$ on arbitrary $\mathfrak{L}_p^2$-algebras, we obtain a 1-1 correspondence of $\mathfrak{N}_p^2$ with $\mathfrak{L}_p^2$ at level of objects.

In addition, Baudisch works in the subclass $\mathfrak{G} = \{G \in \mathfrak{N}_p^2 \mid G' = Z(G)\}$. For if $H \leqslant G$ and both $H, G \in \mathfrak{G}$, then we have an $\mathfrak{L}_p^2$-inclusion of $\mathsf{gr}(H)$ into $\mathsf{gr}(G)$. This is because, the condition in $\mathfrak{G}$ implies $H' = G' \cap H$ and hence H_{ab} embeds as a vector space into G_{ab}.

The class $\mathfrak{G}$ allows therefore to switch between groups and algebras in a clean way when we manipulate group embeddings in the Fraïssé construction.

On the other hand since $Z(G)$ – like every term of the upper central series – is a definable set in the pure group language, properties of $\mathsf{gr}(G)$ may be described at the first order with the signature of groups only.

In Chapter 2 we re-obtain this property for $\mathfrak{L}_p^2$-algebras M, by imposing 2-generated $\mathfrak{L}_p^2$-subalgebras to be free.

If we consider the analogous property in $\mathfrak{N}_p^c$, namely

$$\zeta_k(G) = \gamma_{c+1-k}(G) \quad \text{for } G \in \mathfrak{N}_p^c, \tag{1.21}$$

then the group-algebra correspondences introduced so far, like gr and – reversely – Proposition 1.4.16 and (1.20), all preserve this feature from G to M and vice-versa: (1.21) holds iff $\zeta_k(M) = \gamma_{c+1-k}(M)$ for $M \in \mathfrak{N}_p^c$. Note that in both cases the class c condition, always imply the inclusion $\zeta_k(\,\cdot\,) \supseteq \gamma_{c+1-k}(\,\cdot\,)$.

1.4.1 Deficiency and Group Homology

In this section we see how the second homology of a finitely presented group or Lie algebra, together with the first lower central section, entirely captures the relevant informations expressible in terms of generators and relations.

[4] This is not the Hausdorff formula (1.19), (1.17). To obtain it one has to replace $r_\alpha s_\beta$ with $\frac{1}{2}(r_\alpha s_\beta - r_\beta s_\alpha)$ in the last summand.

The objects we illustrate below present strong similarities with those defined in the second and especially the third chapter. The last were developed independently from appealing to homology.

We refer to the book [HS71] for the basic facts concerning group homology.

A *finitely presented* group G is (n,r)-presented, if it admits a presentation with n generators and r relators. The *deficiency* of G is defined as

$$\mathrm{def}(G) = \max(n - r \mid G \text{ is } (n,r)\text{-presented}).$$

It is possible to estimate the deficiency of a finitely presented group G in terms of the *Schur multiplicator* $H_2(G) = H_2(G,\mathbb{Z})$, the second homology group of G with integer coefficients. The following result, in [Rob96, 14.1.5], is attributed to Philip Hall.

If A is a finitely generated abelian group, we denote by $rk(A)$ the rank of A and $d(A)$ the minimal number of elements required to generate A.

Fact 1.4.23. *If G is a finitely presented group, then $H_2(G)$ is finitely generated. Moreover*

$$\mathrm{def}(G) \leq rk(G_{ab}) - d(H_2(G)). \tag{1.22}$$

Hopf's formula ([Hop42]) expresses $H_2(G)$ in terms of any presentation of the group G:

$$H_2(G) = \frac{F' \cap R}{[F,R]} \tag{1.23}$$

provided $R \to F \to G$ presents G.

Hopf's formula was later recognised independently by Stallings ([Sta65]) and Stammbach ([Sta66],[HS71, §8]) to stem from the *5-term* Homology sequence

$$H_2(E) \to H_2(Q) \to N/[E,N] \to E_{ab} \to Q_{ab} \to \mathbf{0} \tag{1.24}$$

associated to any short exact sequence $N \to E \to Q$, by applying (1.24) to a presentation $R \to F \to G$ of the group G. Now (1.23) follows by the fact $H_2(F) = \mathbf{0}$.

We can specialise – with Stammbach's [Sta73, §III] – the 5-term sequence above to a group variety $\mathcal{V}$, obtaining a notion of schur multiplier $H_2(G;\mathcal{V},B)$ relative to $\mathcal{V}$ for any G-module B and group $G \in \mathcal{V}$. With this technique, an Hopf formula in terms of $\mathcal{V}$-presentations is achieved. On the other hand Stallings ([Sta65, Theorem 2.1]) points out how group homology with coefficients in $\mathbb{Z}/p\mathbb{Z}$, is connected to a *p-exponent modification*[5] of the lower central series.

Inspired by the results cited above, similar features concerning finitely presented $\mathfrak{N}_p^c$-groups may be derived. Note that in general a finitely generated nilpotent group is also finitely presented, being this property closed under extensions of groups.

[5] the group words $\gamma_k(\bar{x})$, which define the lower central series are replaced by $\gamma_k(\bar{x})y^p$. In a group of exponent p we reobtain the old series.

Lemma 1.4.24. *We say that $G \in \mathfrak{N}_p^c$ is* (n,r)-presented in $\mathfrak{N}_p^c$ *if it admits a presentation by the n-generated $\mathfrak{N}_p^c$-free group F modulo a normal subgroup R, which is the normal closure of r elements of F.*

If $\mathrm{def}_{\mathfrak{N}_p^c}(G) = max(n - r \mid G$ *is* (n,r)*-presented in* $\mathfrak{N}_p^c)$*, then this number exists finite and we have*

$$\mathrm{def}_{\mathfrak{N}_p^c}(G) \leq \dim_{\mathbb{Z}_p}(G_{ab}) - \dim_{\mathbb{Z}_p}(H_2(G; \mathfrak{N}_p^c)) \tag{1.25}$$

where $H_2(G; \mathfrak{N}_p^c)$ is defined *as the kernel of the natural map ϕ in the exact sequence of $\mathbb{Z}_p$-vector spaces*

$$R/[F,R] \xrightarrow{\phi} F_{ab} \to F/RF' \to \mathbf{0} \tag{1.26}$$

where $R \to F \to G$ is any finite *$\mathfrak{N}_p^c$-presentation of G.*

Remark. By (1.26) we have

$$H_2(G; \mathfrak{N}_p^c) = \frac{F' \cap R}{[F,R]}$$

and by [Sta73, §III.1,2] this group does not depend of the chosen $\mathfrak{N}_p^c$-presentation.

Proof of Lemma 1.4.24. Assume the group G is (n,r)-presented in $\mathfrak{N}_p^c$ by F modulo R.

Since F is the n-generated $\mathfrak{N}_p^c$-free group, we have $\dim_{\mathbb{Z}_p}(F_{ab}) = n$. Exactness in (1.26) now yields $n - r \leq \dim_{\mathbb{Z}_p}(F_{ab}) - \dim_{\mathbb{Z}_p}(R/[F,R]) = \dim_{\mathbb{Z}_p}(G_{ab}) - \dim_{\mathbb{Z}_p}(H_2(G; \mathfrak{N}_p^c))$.

□

We list some facts to underline the strength of these concepts.

Fact ([Sta65, Theorem 6.5])**.** *Let G be a $\mathfrak{N}_p^c$-group with $H_2(G; \mathfrak{N}_p^c) = \mathbf{0}$ and $(x_i)_{i \in I}$ a set of elements in G whose images in G_{ab} are $\mathbb{Z}_p$-linearly independent. Then the x_i's generate a $\mathfrak{N}_p^c$-free subgroup of G.*

Fact ([Sta65, Sta66])**.** *Let ϕ be a group homomorphism of G in K, if ϕ induces an isomorphism of G_{ab} to K_{ab} and an epimorphism ϕ_* of $H_2(G)$ onto $H_2(K)$, then ϕ induces isomorphisms of $G/\gamma_i(G)$ to $K/\gamma_i(K)$ for all $i < \omega$.*

In particular if G and K are nilpotent, they are isomorphic.

A finitely presented group is called *efficient* if equality holds in (1.22) and $\mathcal{V}$-efficient if the same equality holds for the corresponding $\mathcal{V}$-deficiency.

Fact ([Sta73, Theorem 6.5])**.** *Let G be a group in $\mathcal{V}$, given by a finite $\mathcal{V}$-presentation. Then there exists an efficient group $K \in \mathcal{V}$ and a surjective homomorphism $f\colon K \to G$ which induces an isomorphism $f_i\colon K/\gamma_i(K) \to G/\gamma_i(G)$ for every $i \geq 1$.*

In particular $\mathfrak{N}_p^c$-groups are $\mathfrak{N}_p^c$-efficient: equality in (1.25) holds!

The objects and facts reported above apply, in the very same fashion, to Lie algebras. One may check [KS67] or [HS71, §VII].

In particular for a presentation $\mathfrak{r} \to \mathfrak{f} \to \mathfrak{g}$, the second integral homology group of $\mathfrak{g}$ is given by

$$H_2(\mathfrak{g}) = \mathfrak{f}' \cap \mathfrak{r}/[\mathfrak{f}, \mathfrak{r}]$$

As before, we find the analogous notion related to our special class of Lie algebras $\mathfrak{L}_p^c$ over $\mathbb{Z}_p$. In particular for $M = \langle M_1 \mid R \rangle$ in $\mathfrak{L}_p^c$, as R is contained in the *commutator algebra* $(L^c(M_1))' = \gamma_2(L^c(M_1))$, we find

$$H_2(M, \mathfrak{L}_p^c) = \frac{R}{[R, L]} \tag{1.27}$$

where L denotes $L^c(M_1)$.

Of course we can define – as in Lemma 1.4.24 – the corresponding notion $\mathrm{def}_{\mathfrak{L}_p^c}$ of $\mathfrak{L}_p^c$-deficiency for finitely generated algebras M. We may as well speak of *efficient* $\mathfrak{L}_p^c$-*algebras* and in particular, we have

$$\mathrm{def}_{\mathfrak{L}_p^c}(M) = \dim_{\mathbb{Z}_p}(M_1) - \dim_{\mathbb{Z}_p}(H_2(M, \mathfrak{L}_p^c)). \tag{1.28}$$

In our case recall that the ideal R is homogeneous and $R = R_2 \oplus \cdots \oplus R_c$. Roughly speaking the group $R/[L, R]$ mods out for all $i \leq c$, the relators R_i of weight i of the redundant terms: the elements of R_i which arise as brackets $[r', x_1, \ldots, x_{i-k}]$, for relators of *lower weight* $r' \in R_k$.

In section 3.1 of Chapter 3 we will encounter exactly this *shifting* phenomenon.

It is worth to note that the above notions interact with free products with amalgamated subgroup. This is connected with the Mayer-Vietoris sequence (cfr. [Sta73, §II.6]).

2 Nilpotency Class 2

In this chapter we develop a Fraïssé-Hrushowski construction within the class $\mathfrak{L}_p^2$, described in the previous section. This will lead to the uncollapsed theory T^2 of a rich 2-nilpotent Lie algebra. As pointed out before the prime p has to be chosen greater than 2.

The language $\mathscr{L}^2$ adopted in Section 1.4 contains, along with the Lie ring signature, two predicates P_1 and P_2 to interpret the grading $M = P_1(M) \oplus P_2(M)$ of any $\mathfrak{L}_p^2$-algebra M. As defined in Section 1.4 we have $M = \langle P_1(M) \rangle = \langle M_1 \rangle$ and hence $M_2 = P_2(M)$ is the subspace generated by commutator-lenght 2 elements.

2.1 Deficiency Calculus

Recall that, any object M of $\mathfrak{L}_p^2$ is associated a *presentation*

$$R \longrightarrow L^2(M_1) \longrightarrow M$$

and we write $M = \langle M_1 \mid R \rangle$, where R is an homogeneous ideal of the free nil-2 Lie algebra $L^2(M_1)$, of total weight weight 2. That is, R is a $\mathbb{Z}_p$-vector subspace of $(L^2(M_1))_2$.

We let the homogeneous subspace $(L^2(M_1))_2$ coincide with the exterior square of the $\mathbb{Z}_p$-vector space M_1 and hence $L^2(M_1) \simeq M_1 \oplus \bigwedge^2 M_1$ (see [Ser06, §I.1]).

Hence if M is given by the presentation[1] above, we denote R by $R^2(M)$ and we have

$$M \simeq M_1 \oplus \frac{\bigwedge^2 M_1}{R^2(M)}.$$

If M is an object of $\mathfrak{L}_p^2$, an $\mathfrak{L}_p^2$-*subalgebra* is by definition, a subalgebra H of M, which is generated by a $\mathbb{Z}_p$-subspace H_1 of M_1. We write in this case $H = \langle H_1 \rangle^M$. Conversely for any subspace H_1 of M_1, we adopt the convention to denote by H the $\mathfrak{L}_p^2$-subalgebra $\langle H_1 \rangle^M$. By *subalgebras* we will exclusively mean $\mathfrak{L}_p^2$-subalgebras in the future.

For $\mathfrak{L}_p^2$-subalgebras A and B of M, with abuse of the common meaning we *denote* by $A + B$ the subalgebra $\langle A_1 + B_1 \rangle^M$. As $\mathbb{Z}_p$-vector spaces, *finitely generated* $\mathfrak{L}_p^2$-algebras are *finite*.

For $M \in \mathfrak{L}_p^2$ and a subspace H_1 of M_1 we consider $\bigwedge^2 H_1$ as a natural subspace of $\bigwedge^2 M_1$. To any such H_1 or equivalently, to any $\mathfrak{L}_p^2$-subalgebra $H = \langle H_1 \rangle^M$ of M we set

$$R_M^2(H_1) = R_M^2(H) := R^2(M) \cap \bigwedge\nolimits^2 H_1. \tag{2.1}$$

[1]The couple $\big(M_1, R^2(M_1)\big)$ informally represents the kind of structures utilised in [Bau96].

If the ambient structure M is clear from the context, we simply write $R^2(H_1)$ or $R^2(H)$. In any case we have[2]

$$H \simeq (L^2(H_1) + R^2(M))/R^2(M) \simeq L^2(H_1)/R^2_M(H).$$

We now introduce an integer valued function δ with entries on the finite $\mathbb{Z}_p$-subspaces of M_1, which measures in terms of $\mathbb{Z}_p$-dimension, *how much a finitely generated structure differs from a free one.* The term *deficiency* is also motivated by section 1.4.1, observe in this case $H_2(M) = R^2(M)$.

Definition 2.1.1. Assume an algebra M of $\mathfrak{L}^2_p$ has been fixed. For a finite $\mathfrak{L}^2_p$-subalgebra A of M set

$$\delta(A) = \dim_{\mathbb{Z}_p}(A_1) - \dim_{\mathbb{Z}_p}\left(R^2_M(A)\right). \tag{2.2}$$

We call $\delta(A)$ the *deficiency* of the subalgebra $A = \langle A_1 \rangle^M$ of M.

Observe that if an algebra M is fixed, then $\delta(A)$ depends – by (2.1) – only on the subspace A_1 of M_1. In fact we will write indifferently $\delta(A_1)$ or $\delta(A)$ for the deficiency of A.

Also, $\delta(A)$ is an invariant of the isomorphism type of the structure A and $\delta(A) = \dim_{\mathbb{Z}_p}(A_1)$ implies $A \simeq L^2(A_1)$.

For arbitrary $\mathfrak{L}^2_p$-subalgebras H of M, and finite C_1 (over H_1), we introduce a *relative* deficiency[3] by means of

$$\delta(C/H) = \dim_{\mathbb{Z}_p}(C_1/H_1) - \dim_{\mathbb{Z}_p}(R^2_M(C/H))$$

provided we define $R^2_M(C/H)$ to be the quotient space $R^2(H_1 + C_1)R^2(H_1)$. We also allow expressions $\delta(C_1/H)$ and $\delta(C_1/H_1)$ to denote the above.

For finite A and B, we have of course $\delta(A/B) = \delta(A+B) - \delta(B)$, while for finite *sets* $\mathcal{U}$ or *tuples* in M_1 and arbitrary H, we set $\delta(\mathcal{U}/H) = \delta(\langle H_1, \mathcal{U}\rangle_{\mathbb{Z}_p}/H)$ and $\delta(\bar{a}/H) = \delta(\langle H, \bar{a}\rangle_{\mathbb{Z}_p}/H)$.

Now let M be an $\mathfrak{L}^2_p$-algebra, by virtue of Fact 1.4.10 we have

Remark 2.1.2. For all H_1 and K_1 in M_1

$$\bigwedge^2(H_1 \cap K_1) = \bigwedge^2 H_1 \cap \bigwedge^2 K_1 \tag{2.3}$$

Corollary 2.1.3. *Fixed an algebra M of $\mathfrak{L}^2_p$, we observe a modular behaviour of the operator R^2 on the subspaces of M_1, that is for all H_1 and K_1,*

$$R^2(H_1 \cap K_1) = R^2(H_1) \cap R^2(K_1) \tag{2.4}$$

[2] Here below instead, + indicates the ordinary sum between a subalgebra and an *ideal.* In the sequel this will be almost never the case.

[3] with values in $\mathbb{Z} \cup \{-\infty\}$.

As a first consequence of the above results, we obtain that δ is actually a predimension *on* M_1. In fact the relative deficiency satisfies a stronger version of submodularity.

Lemma 2.1.4. *Let $H_1 \supseteq V_1$ and C_1 be subspaces of M_1, for M in $\mathfrak{L}_p^2$. If C_1 is finite and $H_1 \cap C_1 \subseteq V_1$, then $\delta(C/H) \leq \delta(C/V)$.*

Proof. On one side, the assumption yields $\dim_{\mathbb{Z}_p}(C_1/H_1) = \dim_{\mathbb{Z}_p}(C_1/V_1)$.
For the negative part of δ, observe that $R^2(C/V)$ embeds into $R^2(C/H)$. This follows from

$$R^2(V+C) \cap R^2(H) = R^2((V_1 + C_1) \cap H_1) = R^2(V_1 + (H_1 \cap C_1)) = R^2(V).$$

□

As an extremal case, we get *submodularity* for δ on M_1, that is

$$\delta(C/H) \leq \delta(C_1/C_1 \cap H_1) \tag{2.5}$$

for any H_1 and finite C_1. On finite spaces, if cl denotes the $\mathbb{Z}_p$-linear span in M_1, this is exactly (1.3).

The next obliged step is to force δ to be non-negative. With this purpose define the property

$$(\Sigma^2(2)) \qquad \text{for any finite } A_1 \subseteq M_1,\ \delta(A) \geq \min(2, \dim_{\mathbb{Z}_p}(A_1)).$$

As δ is an invariant of the isomorphism type of finite $\mathfrak{L}_p^2$-algebras, property $\Sigma^2(2)$ is first order expressible in the language $\mathscr{L}^2$ by a denumerable axiom system: just negate the diagrams of those which *do not* have the desired property.

Some remark about the choice of the number 2 as a lower bound are to be given. Of course 1-generated subalgebras are isomorphic to $\mathbb{Z}_p$ in any $\mathfrak{L}_p^c$-algebra.

Condition $\Sigma^2(2)$ imposes that 2-generated subalgebras are free. Equivalently, for any fixed element $a \in M_1$ for M with $\Sigma^2(2)$, the the kernel of the natural derivation $ad_a \colon x \mapsto [a,x]$ coincides with $\langle a \rangle_{\mathbb{Z}_p}$ and, as a consequence the centre $Z(M)$ is forced to coincide with M_2. This last condition – which is equivalent to require the form $[\cdot,\cdot]$ to be non-degenerate – is therefore weaker than $\Sigma^2(2)$.

This feature reflects to the associated group $G(M)$ (or $\mathscr{G}(M)$) reconstructed from M in Section 1.4 (cfr. Remark 1.4.22). In particular, $\mathfrak{N}_p^2$-groups obtained via $G(\cdot)$ from $\mathfrak{L}_p^2$-algebras with $\Sigma^2(2)$ all share the property $G' = Z(G)$.

On the other hand, $\Sigma^2(2)$ influences the pregeometry on M_1 associated to δ (cfr. Corollary (2.1.9) below).

Remark 2.1.5. Assume M has $\Sigma^2(2)$, if cl denotes the $\mathbb{Z}_p$-linear closure in M_1, then δ defines a cl-predimension on M_1 according to definition 1.1.4.

Denote by d^M or simply d, the dimension function on M_1 associated to δ with Lemma 1.1.5 and and by cl_d^M or cl_d the resulting closure. For finite $\mathfrak{L}_p^2$-subalgebras A, we have

- $d(A) := d(A_1) = \min(\delta(C) \mid C_1 \supseteq A_1)$ and $d(A) \leq \dim_{\mathbb{Z}_p}(A_1)$

- cl_d extends cl and $b \in \mathit{cl}_d(A_1)$ exactly if $d(A_1, b) = d(A)$.

In presence of $\Sigma^2(2)$ the notion of *self-sufficiency* which follows, let us *choose* for any given A_1, a distinguished minimal space of deficiency $d(A)$ above A_1.

Definition 2.1.6. Let H_1 be a subspace of M_1, for $M \in \mathfrak{L}_p^2$. We call both H_1 and the $\mathfrak{L}_p^2$-sublagebra H, *strong* or *self-sufficient* in M_1 or M respectively if for any finite subspace $C_1 \subseteq M_1$, we have $\delta(C/H) \geq 0$. This is written $H_1 \leqslant M_1$ or $H \leqslant M$.

For any integer $n < \omega$, we say that H is n-strong in M, if $\delta(C_1/H_1) \geq 0$ holds for all subspaces C_1 of M_1 with $\dim_{\mathbb{Z}_p}(C_1/H_1) \leq n$. We write in this case $H \leqslant^n M$. We say that an $\mathfrak{L}_p^2$-embedding ϕ of H into M is (n-)*strong* if $\phi(H)$ is (n-) strong in M.

Remark 2.1.7. A finite subspace A_1 of M_1 is self-sufficient in M_1 if and only if $d^M(A) = \delta(A)$ and in general $d(A) \leq \delta(A)$.

Definition 2.1.8. Let B_1 be a finite subspace of M_1, define a *self-sufficient closure* of B_1 in M_1 to be an $\subseteq$-minimal subspace A_1 of M_1 containing B_1 with $\delta(A) = d(B)$. By Lemma 2.1.13 below, the family of strong subspaces of M_1 is closed under intersection, as a consequence the notion of self-sufficient closure of a finite space A_1 depends on A and M only and is univocally determined as:

$$ssc^M(A_1) = ssc(A_1) := \bigcap\{C_1 \leqslant M_1 \mid C_1 \text{ finite and } C_1 \supseteq A_1\}$$

We define the $\mathfrak{L}_p^2$-subalgebra $ssc^M(A) = ssc(A)$ of M as $\langle ssc(H_1)\rangle^M$and we call it the self-sufficient closure *of* H in M. For a finite *subset* $\mathcal{U}$ of M_1, we set $ssc(\mathcal{U})$ to be $ssc(\langle\mathcal{U}\rangle_{\mathbb{Z}_p})$.

Note that this definition implies the operator ssc is actually a closure operator: it is monotone, and has properties (cl1) and (cl2) of definition 1.1.1, moreover $ssc(A_1) \subseteq \mathit{cl}_d(A_1)$.

Corollary 2.1.9. *For any algebra M with $\Sigma^2(2)$, the pregeometry (M_1, cl_d) associated to δ is actually a geometry over the $\mathbb{Z}_p$-linear closure according to Definition 1.1.3.*

For a given M, the geometry cl_d is not in general locally-modular.

Proof. Axiom $\Sigma^2(2)$ implies any two linearly independent couple a, b generates a self-sufficient subalgebra $\langle a, b\rangle^M \simeq L^2(a, b)$ and $d(a, b) = 2$. Analogously $d(\langle\varnothing\rangle_{\mathbb{Z}_p}) = d(\mathbf{0}) = \delta(\mathbf{0}) = 0$ and for any $a \in M_1$, $d(\langle a\rangle_{\mathbb{Z}_p}) = \delta(a) = 1$. It follows $\mathit{cl}_d(\varnothing) = \mathbf{0}$ and $\mathit{cl}_d(a) = \langle a\rangle_{\mathbb{Z}_p}$.

Consider now the finite algebra $M = \langle M_1 \mid R^2(M)\rangle$ whose M_1 has $\mathbb{Z}_p$-base $\{a, b, c, x, y\}$ and such that the relator ideal $R^2(M)$ is spanned in $\bigwedge^2 M_1$ by the independent homogeneous elements

$$[a, b] + [x, y], \quad [c, x] + [y, b], \quad [a, y] + [b, c].$$

One checks that M has $\Sigma^2(2)$ and that the subspace $\langle a, b, c\rangle_{\mathbb{Z}_p}$ is not self-sufficient in M: in fact $\delta(M/a, b, c) = -1$. It follows $2 = d(a, b, c) < \delta(a, b, c) = 3$ and this yields

$$d(a, b) + d(b, c) = 4 > 3 = d(a, b, c) + d(b).$$

(1.2) is not (even locally) satisfied.

□

Some properties of sef-sufficient spaces will now follow. We assume an algebra M of $\mathfrak{L}_p^2$ has been fixed with $\Sigma^2(2)$. All the subspaces and subalgebras considered, lay in M_1 and M respectively.

Remark 2.1.10. For a self-sufficient H and a finite A, we can always find a *finite* strong subalgebra H^{o} such that $\delta(A/H) = \delta(A/H^{\mathrm{o}})$.

For an arbitrary H, one has

$$\delta(A/H) = \inf\left(\delta(A/C) \mid C_1 \text{ finite and } A_1 \cap H_1 \subseteq C_1 \leqslant H_1\right).$$

Proof. For the first part, since $R^2(A/H)$ has to be finite dimensional, pick a finite $\mathfrak{L}_p^2$-subalgebra H^{o} in H with $H_1^{\mathrm{o}} \supseteq H_1 \cap A_1$ and such that $R^2(H + A)$ has a basis in $\bigwedge^2 \langle H_1^{\mathrm{o}}, A_1\rangle_{\mathbb{Z}_p}$ over $\bigwedge^2 H_1$. By Corollary 2.1.13 below we can choose H^{o} to be self-sufficient.

The second part follows by Lemma 2.1.4 and the above arguments.

□

The next lemma shows *transitivity* of strong embeddings.

Lemma 2.1.11. *If H is n-strong in K and K is self-sufficient in M, then H is n-strong in M.*

In particular from $H \leqslant K$ and $K \leqslant M$, follows $H \leqslant M$.

Proof. Let C_1 be a finite subspace, both statements of the lemma follow from the inequality $\delta(C_1/H) \geq \delta(C_1 \cap K_1/H) + \delta(C_1/K)$.

We have equality for the $\mathbb{Z}_p$-linear dimensions and for the negative parts, we observe that $R^2(C/H)$ maps to $R^2(C/K)$ with kernel

$$\frac{R^2(H+C) \cap R^2(K)}{R^2(H)} = \frac{R^2((H_1 + C_1) \cap K_1)}{R^2(H)} = \frac{R^2(H_1 + (C_1 \cap K_1))}{R^2(H)}.$$

□

Another straightforward application of lemma 2.1.4 is the following:

Lemma 2.1.12 (Cut Lemma)**.** *If H is self-sufficient in K, then for any subspace V_1 of M_1, we have $H_1 \cap V_1 \leqslant K_1 \cap V_1$.*

Proof. Observe $\delta(E_1/H_1 \cap V_1) \geq \delta(E/H) \geq 0$ whenever $E_1 \subseteq K_1 \cap V_1$, since $H_1 \cap V_1$ contains $E_1 \cap H_1$.

□

Corollary 2.1.13. *If H and K are self-sufficient, then the intersection $H_1 \cap K_1$ is also strong in M_1.*

Proof. By Lemma 2.1.12 we have $H_1 \cap K_1 \leqslant K_1$. Then conclude by transitivity of $\leqslant$ (Lemma 2.1.11).

$\square$

Lemma 2.1.14. *Let H_1 be a subspace of M_1 then H is strong if and only if for any finite subspace C_1 of H_1 there exists a finite subspace $C_1^{\mathrm{o}} \subseteq H_1$ containing C_1, such that $C^{\mathrm{o}} \leqslant M$.*

Proof. If H is strong, given any finite C_1 in H_1, then take C^{o} to be $ssc(C)$. Because of Lemma 2.1.13 C^{o} is contained in H.

For the converse, if A_1 is finite in M_1, we want $\delta(A/H)$ to be non negative. But this follows by the hypothesis applying Remark 2.1.10.

$\square$

Given two algebras $N \subseteq M$ of $\mathfrak{L}_p^2$ the self-sufficient closure of a finite subspace A_1 of N_1 computed in N may differ from $ssc^M(A_1)$. But as expected we have

Remark 2.1.15. Assume N is an $\mathfrak{L}_p^2$-subalgebra of M. Then N is strong in M if and only if for all subspaces V_1 of N_1 the closures $ssc^N(V)$ and $ssc^M(V)$ coincide.

Proof. We may suppose V are *finite* subalgebras in the statement and in general $ssc^M(V) \subseteq ssc^N(V)$ as strongness is expressible via universal sentences.

The first condition is clearly sufficient. It is necessary by virtue of Lemma 2.1.14, since for any finite $V_1 \subseteq N_1$, we now know $ssc^N(V) = ssc^M(V)$ is *inside* N, but strong *in* M.

$\square$

We might have stated Remark 2.1.15 in terms of $c\ell_d$-dimensions: $N \leqslant M \iff d^N(V_1) = d^M(V_1)$ for any subspace $V_1 \subseteq N_1$.

Lemma 2.1.16. *Assume $H \leqslant M$ and $\delta(A/H) = 0$ for some finite subspace A_1 of M_1, then $H + A$ is self-sufficient in M as well.*

Moreover if an element a of M_1 is $c\ell_d$-independent of H_1, i.e. $d^M(a/H) = 1$, then $\langle H_1, a\rangle^M$ is strong in M.

Proof. Consider a finite subspace E_1 of M_1, then the first statement follows by computing

$$\delta(E/H + A) = \delta(E + A/H) - \delta(A/H).$$

For the second one, note that any finite subspace of $\langle H_1, a\rangle_{\mathbb{Z}_p}$ is contained in some $\langle A_1, a\rangle_{\mathbb{Z}_p}$ where A_1 is a finite strong subspace of H_1. Since $d = d^M$ is a dimension, $1 = d(a/H_1) \leq d(a/A_1) \leq 1$ implies $d(A_1) + 1 = d(A_1, a)$.

We conclude by Lemma 2.1.14 showing that $\langle A_1, a\rangle_{\mathbb{Z}_p} \leqslant M_1$. We have indeed, since $A \leqslant M$, $\delta(A_1, a) \leq \delta(A) + 1 = d(A_1) + 1 = d(A_1, a)$. This yields $\delta(A_1, a) = d(A_1, a)$.

$\square$

For an arbitrary space H_1 we define the *self-sufficient closure* $ssc^M(H_1)$ of H_1 as the subspace of M_1 generated by the self-sufficient closures of all the finite parts of H_1. This space is strong on account of Lemma 2.1.14. As before, by $ssc(H)$ we mean $\langle ssc(H_1)\rangle^M$. This is the minimal strong $\mathfrak{L}_p^2$-subalgebra of M containig H.

We adopt for the sequel the following notation: for any subspaceH_1 and tuple $\bar{a}$ of M_1, we write $ssc(H_1, \bar{a})$ for $ssc(\langle H_1, \bar{a}\rangle_{\mathbb{Z}_p})$ and $ssc(H, \bar{a})$ for $\langle ssc(H_1, \bar{a})\rangle^M$. On the other hand, by default d^M reads indifferently sets, subspaces or tuples of M_1.

Proposition 2.1.17. *Assume H_1 is a strong subspace of M_1 and $\bar{a}$ is a finite tuple in M_1, then*

(i) $d(\bar{a}/H_1) \leq \delta(\bar{a}/H_1)$

(ii) *$ssc(H_1, \bar{a})$ is a* finite *extension of H_1,*

(iii) $d(\bar{a}/H_1) = \delta(ssc(H_1, \bar{a})/H_1) = \min(\delta(A_1/H_1) \mid A_1 \supseteq \bar{a})$

(iv) $d(\bar{a}/H_1) = \delta(\bar{a}/H_1)$ *iff* $\langle H_1, \bar{a}\rangle_{\mathbb{Z}_p} \leqslant M_1$.

(v) *There exists a finite $H_1^{\mathrm{o}} \leqslant H_1$ such that $ssc(H, \bar{a}) = H + ssc(H^{\mathrm{o}}, \bar{a})$, $H_1 \cap ssc(H_1^{\mathrm{o}}, \bar{a}) = H_1^{\mathrm{o}}$ and that $d(\bar{a}/H) = d(\bar{a}/H^{\mathrm{o}})$.*

Proof. (i). By (fin) and (1.1) of Section 1.1 and Remark 2.1.10 above, we can find a finite subspace $H_1^{\mathrm{o}} \leqslant H_1$ with $H_1^{\mathrm{o}} \supseteq H_1 \cap \langle \bar{a}\rangle_{\mathbb{Z}_p}$, such that

$$d(\bar{a}/H_1) = d(\bar{a}/H_1^{\mathrm{o}}) = d(H_1^{\mathrm{o}}, \bar{a}) - d(H_1^{\mathrm{o}})$$

and that

$$\delta(\bar{a}/H_1) = \delta(\bar{a}/H_1^{\mathrm{o}}) = \delta(H_1^{\mathrm{o}}, \bar{a}) - \delta(H_1^{\mathrm{o}}).$$

Since $H^{\mathrm{o}} \leqslant M$, the statement follows immediately by the relation between δ and d for finite subspaces of M_1.

(ii). Since $\delta(A/H)$ is non-negative for all finite subspace A_1 in M_1, take a finite subspace A_1 containing $\bar{a}$ with a minimal value of $\delta(A/H)$. It follows that for an arbitrary finite C_1 one has

$$\delta(C/H + A) = \delta(C + A/H) - \delta(A/H) \geq 0.$$

This means $H + A$ is self-sufficient in M and hence contains $ssc(H, \bar{a})$.

As a consequence, the second equality in (iii) holds: $\delta(ssc(H_1, \bar{a})/H) = \min(\delta(A/H) \mid \bar{a} \subseteq A_1 \subseteq M_1, A_1 \text{ finite})$.

(iii). Take a finite tuple $\bar{b}$ of M_1 linear independent over H_1, such that $\langle H_1, \bar{b}\rangle_{\mathbb{Z}_p} = ssc(H_1, \bar{a})$. Since $\bar{b} \subseteq ssc(H_1, \bar{a}) \subseteq c\ell_d(\bar{a}/H_1)$, we have $d(\bar{b}/H_1) = d(\bar{a}/H_1)$.

As $\langle H_1, \bar{b}\rangle_{\mathbb{Z}_p}$ is self-sufficient, we can find a finite strong subalgebra H^{o} of H, such that $\langle H_1^{\mathrm{o}}, \bar{b}\rangle_{\mathbb{Z}_p} \leqslant M_1$ with $H_1^{\mathrm{o}} \supseteq H_1 \cap \langle \bar{a}\rangle_{\mathbb{Z}_p}$ and $d(\bar{b}/H^{\mathrm{o}}) = d(\bar{b}/H)$.

Now by (i) and Lemma 2.1.4 we obtain $d(\bar{b}/H) \leq \delta(\bar{b}/H) \leq \delta(\bar{b}/H^{\mathrm{o}}) = d(\bar{b}/H^{\mathrm{o}})$ and hence $\delta(ssc(H_1, \bar{a})/H) = \delta(\bar{b}/H) = d(\bar{a}/H)$.

(iv). Follows from (iii).

(v). Let H^o and $\bar{b}$ like in (iii). above, since $ssc(H_1^o, \bar{a}) \subseteq \langle H_1^o, \bar{b} \rangle_{\mathbb{Z}_p}$, we have

$$H_1^o = \langle H_1^o, \bar{b} \rangle_{\mathbb{Z}_p} \cap H_1 \supseteq ssc(H_1^o, \bar{a}) \cap H_1 \supseteq H_1^o$$

that is $H_1^o = ssc(H_1^o, \bar{a}) \cap H_1$.

On the other hand, by applying submodularity (2.5) and (iii) above, we get

$$\delta(\bar{b}/H) \leq \delta(ssc(H_1^o, \bar{a})/H_1) \leq \delta(ssc(H_1^o, \bar{a})/H_1^o) = d(\bar{b}/H^o) = d(\bar{b}/H).$$

Thus by (iv), since $\delta(ssc(H_1^o, \bar{a})/H) = d(\bar{a}/H) = d(ssc(H_1^o, \bar{a})/H)$, we obtain $H + ssc(H^o, \bar{a}) \leqslant M$. It follows $\langle H_1, \bar{b} \rangle_{\mathbb{Z}_p} \subseteq H_1 + ssc(H_1^o, \bar{a})$ and hence $ssc(H_1, \bar{a}) = H_1 + ssc(H_1^o, \bar{a})$. Moreover this yields also $\langle H_1^o, \bar{b} \rangle_{\mathbb{Z}_p} = ssc(H_1^o, \bar{a})$.

□

From the last proposition it follows, for H and $\bar{a}$ as above, that $ssc(H_1, \bar{a})$ is the intersection of all strong subspaces of M_1 containing $\langle H_1, \bar{a} \rangle_{\mathbb{Z}_p}$.

Remark 2.1.18. Let $\mathcal{B}$ be any set of M_1, then $c\ell_d(\mathcal{B})$ is the subspace of M_1 generated by all finite $C_1 \subseteq M_1$ such that $\delta(C_1/\, ssc(\mathcal{B})) = 0$.

In particular $ssc(\mathcal{B}) \subseteq c\ell_d(\mathcal{B})$ for all sets $\mathcal{B}$.

2.2 Amalgamation Results

Denote by $\mathcal{K}_2$ the class of all finitely generated – or equivalently, finite – Lie algebras M in $\mathfrak{L}_p^2$, which share property $\Sigma^2(2)$ defined at page 37. Then $\mathcal{K}_2$ is a denumerable set.

At the end of this section we show properties (HP), (JEP) and (AP) for the class $\mathcal{K}_2$ with respect to strong $\mathfrak{L}_p^2$-embeddings as described in Remark 1.2.3.[4] The proof of Fact 1.2.2, to achieve a countable Fraïssé limit of $(\mathcal{K}_2, \leqslant)$ applies in this case as well and yields the same results.

In accordance to this, we rename by âge(K) the collection of all finite $\mathfrak{L}_p^2$-subalgebras of an algebra K from $\mathfrak{L}_p^2$. If $\widetilde{\mathcal{K}}_2$ denotes the family of all K in $\mathfrak{L}_p^2$ with âge$(K) \subseteq \mathcal{K}_2$, then $\widetilde{\mathcal{K}}_2$ is *almost* an elementary class[5] and we have $\widetilde{\mathcal{K}}_2 = \{K \in \mathfrak{L}_p^2 \mid K \models \Sigma^2(2)\}$.

We say that $H \in \widetilde{\mathcal{K}}_2$ is a *finite* extension of $K = \langle K_1 \rangle^H$ if H_1 has finite $\mathbb{Z}_p$-dimension over K_1 and H is a *strong extension* of K if $K \leqslant H$.

Definition 2.2.1. Let M, N and K be algebras of $\mathfrak{L}_p^2$. Assume we have $\mathfrak{L}_p^2$-embeddings ϕ of N into M and ν of N into K. We say that an $\mathfrak{L}_p^2$-algebra H *amalgamates M and K over N* if there exist $\mathfrak{L}_p^2$-embeddings μ of M into H and ψ of K into H such that $\phi\mu = \nu\psi$. In this case we draw the following square.

[4] We tacitly perform two modifications of the standard method: one changes $\mathscr{L}^2$-embeddings into $\mathfrak{L}_p^2$-ones, the second introduces strongness.

[5] $\mathfrak{L}_p^2$ itself is not elementary as pointed out in Section 1.4, but as a consequence of *richness*, the theory of the Fraïssé limit in the next Section can express property 3. of Definition 1.4.8

$$\begin{tikzcd}
 & H & \\
M \arrow[ur, "\mu"] & & K \arrow[ul, "\psi"'] \\
 & N \arrow[ul, "\phi"] \arrow[ur, "\nu"'] &
\end{tikzcd} \tag{2.6}$$

It is always possible to build amalgams inside $\mathfrak{L}_p^2$ as follows: assume M, N and K as above, we may consider N, without loss of generality, as a common $\mathfrak{L}_p^2$-subalgebra of M and K, that is $\langle N_1\rangle^M = N = \langle N_1\rangle^K$.

We first build the $\mathbb{Z}_p$-vector space amalgam $H_1 = M_1 \oplus_{N_1} K_1$, which is by definition $M_1 \oplus K_1/\Delta(N_1)$ where $\Delta(N_1) = \{(h, -h) \mid h \in N_1\}$. In H_1, M_1 and K_1 meet exactly in N_1.

We now define the *free amalgam of M and K over N* by

$$M \circledast_N K := \frac{L^2(H_1)}{R^2(M) + R^2(N)} = H_1 \oplus \frac{\bigwedge^2 H_1}{R^2(M) + R^2(K)}. \tag{2.7}$$

By a matter of weight $R^2(M \circledast_N K) = R^2(M) + R^2(K)$ is an ideal of $L^2(H_1)$ and hence the definition above is sound. Moreover $M \circledast_N K = \langle H_1\rangle$ and lays in $\mathfrak{L}_p^2$ and $R^2(M) \cap R^2(K) = R^2(N)$.

Remark 2.2.2. $M \circledast_N K$ fits in the diagram (2.6) in the place of H, with the natural $\mathfrak{L}_p^2$-embeddings. That is $M \circledast_N K$ amalgamates M and K over N.

Moreover $M \cap K = \langle M_1\rangle^{M\circledast_N K} \cap \langle K_1\rangle^{M\circledast_N K} = \langle M_1 \cap K_1\rangle^{M\circledast_N K} = N$.

Proof. Let H denote $M \circledast_N K$. Since we identify $L^2(M_1)$ with an $\mathfrak{L}_p^2$-subalgebra of $L^2(H_1)$, we have to show $R^2(H) \cap L^2(M_1) = R^2(M)$, so that the map $w + R^2(M) \mapsto w + R^2(H)$ yields the desired $\mathfrak{L}_p^2$-embedding μ. But this holds, since $(R^2(M) + R^2(K)) \cap L^2(M_1) = (R^2(M) + R^2(K)) \cap \bigwedge^2 M_1 = R^2(M) + (R^2(K) \cap \bigwedge^2 M_1) = R^2(M) + (R^2(K) \cap \bigwedge^2 N_1) = R^2(M) + R^2(N) = R^2(M)$.

A symmetric argument for K implies the statement and the *moreover* part follows by $R^2(H) \subseteq \bigwedge^2 M_1 + \bigwedge^2 K_1$ and $\bigwedge^2 M_1 \cap \bigwedge^2 K_1 = \bigwedge^2 N_1$.

□

The following definitions provides a notion of *inner* free amalgam.

Definition 2.2.3. Assume M and K are $\mathfrak{L}_p^2$-extension of N in a $\widetilde{\mathcal{K}}_2$-algebra H. We say that M is in *free composition with K over N in H* if $M + K (= \langle M_1 + K_1\rangle^H)$ is isomorphic with $M \circledast_N K$. This is equivalent to require that $M_1 \cap K_1 = N_1$ and that

$$R_H^2(H_1 + M_1) \simeq_{\mathbb{Z}_p} R_H^2(M_1) + R_H^2(K_1).$$

Deficiency calculus yields an easy criterion to check for free-compositions:

Lemma 2.2.4. *Let M, N and K be $\mathfrak{L}_p^2$-subalgebras of $H \in \widetilde{\mathcal{K}}_2$, then the following conditions are equivalent:*

(i) *M is in free-composition with K over N,*

(ii) *$M_1 \cap K_1 = N_1$ and $\delta(A/N) = \delta(A/K)$ for any finite subspace A_1 of M_1.*

Proof. If $M_1 \cap K_1 = N_1$ holds, then $\dim_{\mathbb{Z}_p}(M_1/N_1) = \dim_{\mathbb{Z}_p}(M_1/K_1)$ and by (2.3) we have $R^2(M) \cap R^2(K) = R^2(M_1 \cap K_1) = R^2(N)$.

One has in fact a canonical linear embedding of $R^2(M/N)$ into $R^2(M/K)$. This embedding is onto if and only if $R^2(K+M) = R^2(K) + R^2(M)$ but also iff for any finite $A_1 \subseteq M_1$ the corresponding mapping of $R^2(A/N)$ in $R^2(A/K)$ is onto. This is true exactly if $\dim_{\mathbb{Z}_p}(R^2(A/N)) = \dim_{\mathbb{Z}_p}(R^2(A/K))$ and hence exactly when $\delta(A/N) = \delta(A/K)$ for all finite subspaces $A_1 \subseteq M_1$.

□

Remark 2.2.5. For the free amalgam $M \circledast_N K$ with a finite dimensional *side* M_1/N_1, one has $\delta(M/N) = \delta(M/K)$.

At the end of the chapter we will see that *composing* free-compositions turns out to be *transitivity* of forking in the theory of the $\mathcal{K}_2$-rich structure. This lemma will be helping.

Lemma 2.2.6. *Assume $H \supseteq M \supseteq N \subseteq K$ are $\mathfrak{L}_p^2$-extensions. Then*

$$H \circledast_N K \simeq H \circledast_M (M \circledast_N K).$$

Proof. The statement essentially follows because it is true of vector space amalgams, that is $H_1 \oplus_{N_1} K_1 \simeq_{\mathbb{Z}_p} H_1 \oplus_{M_1} (M_1 \oplus_{N_1} K_1)$.

Since $M \circledast_N K$ $\mathfrak{L}_p^2$-embeds into $H \circledast_N K$, to conclude we have to show that H is in free composition with $M \circledast_N K$ over M in $H \circledast_N K$. If deficiencies are computed inside $H \circledast_N K$, we have in fact

$$\begin{aligned}\delta(H/M+K) = \delta(H+M/K) - \delta(M/K) = \delta(H/K) - \delta(M/N) = \\ = \delta(H/N) - \delta(M/N) = \delta(H/M).\end{aligned}$$

Now Lemma 2.2.4 applies.

□

Remark. $H = M \circledast_N K$ with the morphism in (2.6), represents the *amalgamated coproduct* in the category $\mathfrak{L}_p^2$. This means, for any Z in $\mathfrak{L}_p^2$ and $\mathfrak{L}_p^2$-morphisms $\alpha\colon M \to Z$ and $\beta\colon K \to Z$ with $\phi\alpha = \nu\beta$, there exists a unique morphism $\zeta\colon M \circledast_N K \to Z$ with $\mu\zeta = \alpha$ and $\psi\zeta = \beta$.

The next lemma shows that the free amalgam (2.7) preserves self-sufficient extensions.

Lemma 2.2.7. *In the notation of Definition 2.1.6, for any $k < \omega$, $N \leqslant^k K$ holds if and only if $M \leqslant^k M \circledast_N K$ does. In particular $N \leqslant K$ iff $M \leqslant M \circledast_N K$.*

Proof. Consider a subspace $D_1 \supseteq M_1$ of $M_1 \oplus_{N_1} K_1$. Since $D_1 = M_1 + (D_1 \cap K_1)$ one has $D_1/M_1 \simeq_{\mathbb{Z}_p} D_1 \cap K_1/N_1$.

On the other hand, since $R^2(K) = R^2(M \circledast_N K) \cap \bigwedge^2 K_1$, we have by (2.3), $R^2(D) = (R^2(M) + R^2(K)) \cap \bigwedge^2 D_1 = R^2(M) + R^2(D_1 \cap K_1)$. Now this yields $R^2(D/M) = R^2(D_1 \cap K_1/N_1)$.

Hence we may conclude

$$\delta(D/M) = \delta(D_1 \cap K_1/N_1) \tag{2.8}$$

and the statement of the lemma follows.

□

Corollary 2.2.8. *Let $M \supseteq N \subseteq K$ as in the previous lemma. If A denotes $M \circledast_N K$ and $A_1 \supseteq D_1 \supseteq M_1$, let D be $\langle D_1 \rangle^A$ and I denote $\langle D_1 \cap K_1 \rangle^A$. Then*

$$D \simeq M \circledast_N \langle D_1 \cap K_1 \rangle^K \quad \textit{and} \quad A \simeq D \circledast_I K.$$

Proof. We assume for simplicity, that D_1 is finite over M_1. As observed above $D_1 = M_1 + (D_1 \cap K_1)$ and by (2.8) we have

$$\delta(D_1 \cap K_1/M_1) = \delta(D/M) = \delta(D_1 \cap K_1/N_1)$$

and thus, Lemma 2.2.4 gives the first statement. The second follows by the facts $A_1 \simeq_{\mathbb{Z}_p} D_1 \oplus_{I_1} K_1$ and $R^2(A) = R^2(M) + R^2(K) = R^2(D) + R^2(K)$.

□

We now introduce *minimal strong extensions* of $\widetilde{\mathcal{K}}_2$-algebras. This is the main tool to compute the rank of types in the rich $\widetilde{\mathcal{K}}_2$-structures.

Definition 2.2.9. We say that a proper strong $\mathfrak{L}_p^2$-extension $K \leqslant H$ is *minimal* if there is no subspace V_1 strictly in-between H_1 and K_1 such that V is strong in H.

By Lemma 2.1.17,(ii) minimal extensions are necessarily finite, moreover a finite extension H of K is minimal exactly if $\delta(H/K') < 0$ for all $K_1 \subsetneq K_1' \subsetneq H_1$.

It turns out that there are only three types of minimal strong extensions, this is the content of the next proposition

Proposition 2.2.10. *Assume H is a minimal extension of K, then only one of the following three situation may occur.*

(i) *H is a* free *or* transcendental *extension of K, that is* $\dim_{\mathbb{Z}_p}(H_1/K_1) = \delta(H/K) = 1$ *and $R^2(H) = R^2(K)$.*

(ii) *H is an* algebraic *extension of K:* $\dim_{\mathbb{Z}_p}(H_1/K_1) = 1$ *and* $\delta(H/K) = 0$.

(iii) *H is a* prealgebraic *extension of K:* $\dim_{\mathbb{Z}_p}(H_1/K_1) \geq 2$, $\delta(H/K) = 0$ *and for any finite $E_1 \subsetneq H_1$ not entirely contained in K_1 one has* $\delta(E/K) > 0$.

Proof. We know H_1 is finite over K_1 and assume first $d^H(H_1/K_1) = \delta(H/K) = 0$.
If $\delta(h/K) = 0$ for some h in H_1, then $\langle K_1, h\rangle$ is strong in H by Lemma 2.1.16 and by minimality $H = \langle K_1, h\rangle^H$. We are in (ii).

If there is no h with "saldo null"over K, then $\dim_{\mathbb{Z}_p}(H_1/K_1) > 1$ and by minimality for any proper subspace E_1 of H_1 not entirely contained in K_1, we must have $\delta(E/K) > 0$. This gives a prealgebraic extension.

On the other hand if $d^H(H/K) > 0$, then there must be an a of H_1 $\mathcal{cl}_d$-independent of K_1. This implies $d^H(a/K) = 1$ and $\langle K_1, a\rangle^H \leqslant H$, hence $H = \langle K_1, a\rangle^H$ and also $\delta(a/K) = 1$. In particular $R^2(H) = R^2(K)$.

□

An algebraic extension is associated to a divisor element according to the following remark.

Remark 2.2.11. For any $\mathfrak{L}_p^2$-subalgebra K of $H \in \widetilde{\mathcal{K}}_2$, a *divisor* of K is an element a of $H_1 \setminus K_1$ with $\delta(a/M) \leq 0$. This is equivalent to require, that $[a, x] \in K_2$ for some non-trivial element x of K_1. If in addition K is (1-)strong in H and a is a divisor of K, then

$$R_H^2(K_1, a) = R^2(K) \oplus \langle [a, x] - \kappa\rangle_{\mathbb{Z}_p}$$

for some $x \in K_1$ and $\kappa \in \bigwedge^2 K_1$. In particular $\langle K, a\rangle$ is a minimal algebraic extension of K.

Remark 2.2.12.

1. For any $B \in \widetilde{\mathcal{K}}_2$, any fixed b of B_1 and w in B_2, assume there is no $x \in B_1$ with $[x, b] = w$, then there is a minimal algebraic strong extension $A = \langle B_1, a\rangle$ of B in $\widetilde{\mathcal{K}}_2$ such that $[a, b] = w$.

2. For any positive integer n and any $M \in \mathcal{K}_2$, if $\dim_{\mathbb{Z}_p}(M_1)$ is large enough ($\geq 2+2n$), it is possible to find a chain

$$M \leqslant M^1 \leqslant M^2 \leqslant \ldots \leqslant M^n$$

in which M^{i+1} is a minimal prealgebraic extension of M^i for all i and M^n is in $\mathcal{K}_2$.

Proof. 1. Define an extension A of B as follows: set first $A_1 := B_1 \oplus \mathbb{Z}_p$ and let $a \in A_1$ generate A_1 over B_1. Then set $R^2(A) := R^2(B) \oplus \langle [a, b] - \beta\rangle$, where β is an element of $\bigwedge^2 B_1$ which represents w modulo $R^2(B)$. Hence $[a, b] - \beta$ is an element of $\bigwedge^2 A_1$.
Since in $\delta(A/B) = 0$, B is self-sufficient in A. We show next that A is in $\widetilde{\mathcal{K}}_2$. Let E_1 be a finite subspace of A_1, then E_1 has dimension at most 1 over $E_1 \cap B_1$. Thus in a nontrivial case, there exists $b' \in B_1$ such that $E_1 = \langle a + b', E_1 \cap B_1\rangle$.

As by submodularity (2.5) $\delta(E) = \delta(E_1 \cap B_1) + \delta(a + b'/E_1 \cap B_1)$ and by Lemma 2.1.12 $E_1 \cap B_1 \leqslant E_1$, if $\dim_{\mathbb{Z}_p}(E_1 \cap B_1) \geq 2$ we have $\delta(E) \geq 2$.

The other only case to be considered is when $\dim_{\mathbb{Z}_p}(E_1) = 2$ and $E_1 = \langle a + b', u\rangle$ for b', u in B_1. If $R^2(E) \neq \mathbf{0}$, then we may assume the equality $[a + b', u] = [a, b] - \beta + \eta$ holds in $\bigwedge^2 A_1$, for some η in $R^2(B)$. This translates into $[a, u-b] = [u, b'] - \beta + \eta \in \bigwedge^2 B_1$.

If we take any $\mathbb{Z}_p$-basis $(b_i \mid i < n)$ of B_1 for some $n < \omega$, then the set $([a, b_i] \mid i < n)$ is a basis for $\bigwedge^2 A_1$ *over* $\bigwedge^2 B_1$ (cfr. Fact 1.4.10). This yields that $u = b$ and that $[u, b'] - \beta$ belongs to $R^2(B)$. Thus the element $-b'$ of B_1 solves the equation $[-b', b] = w$ in B, contradicting our assumption.

For 2. it is sufficient to prove the first step, assume hence M is in $\mathcal{K}_2$ with at least four linearly independent element b_1, b_2, c_1, c_2 in M_1.

Define an $\mathfrak{L}_p^2$-algebra K by means of the following presentation

$$K = \langle a_1, a_2, M_1 \mid [a_1, b_1] + [a_2, b_2], [a_1, c_1] + [a_2, c_2]\rangle.$$

It is clear that $\delta(K/M) = 0$ and that K is a prealgebraic strong extension of M.

We have to show that K lays in $\mathcal{K}_2$ as well: for any finite $E_1 \subseteq K_1$ we must prove $\delta(E) \geq \min(2, \dim_{\mathbb{Z}_p}(E_1))$.

By (2.5) we have $\delta(E) \geq \delta(E_1 \cap M_1) + \delta(E/M)$. Moreover, for any element u of $K_1 \setminus M_1$, then $\delta(u/M) > 0$. For, since u is without loss $sa_1 + ta_2$ for some $s, t \in \mathbb{Z}_p$, if an element ρ of $R^2(K)$ lays in $\bigwedge^2 \langle M_1, u\rangle_{\mathbb{Z}_p}$, then

$$\rho = [sa_1 + ta_2, m] + \mu = u([a_1, b_1] + [a_2, b_2]) + v([a_1, c_1] + [a_2, c_2]) + \eta$$

for some $u, v \in \mathbb{Z}_p$, $m \in M_1$, $\eta \in R^2(M)$ and some $\mu \in \bigwedge^2 M_1$.

As a consequence we obtain

$$[a_1, sm - ub_1 - vc_1] + [a_2, tm - ub_2 - vc_2] \in \bigwedge^2 M_1 \tag{2.9}$$

which is impossible unless b_1, b_2, c_1, c_2 are linearly dependent.

Now if every 2-generated $\mathfrak{L}_p^2$-subalgebra of K is free, then the same is true of all its 3-generated subalgebras.[6]

Hence by the above inequality, since M has $\Sigma^2(2)$ we only have to prove this property in the case $\dim_{\mathbb{Z}_p}(E_1) = 2$ and $E_1 \cap M_1 = \mathbf{0}$. In which case, with no loss of generality $E_1 = \langle a_1 + l, a_2 + m\rangle_{\mathbb{Z}_p}$ for l, m elements of M_1. Now if some element of $\bigwedge^2 E_1$ meets $R^2(K)$, then a contradiction like (2.9) would follow. Thus $R^2(E) = \mathbf{0}$ in this case and $\Sigma^2(2)$ holds in general for K.

$\square$

[6] Assume a subalgebra is generated by independent elements u, v, z. A non-trivial relator of weight 2 has the form $\alpha[u, v] + [\beta u + \gamma v, z]$ for $\alpha, \beta, \gamma \in \mathbb{Z}_p$ and we may assume $\beta \neq 0$. The above element is also equal to $[\beta u + \gamma v, \alpha\beta^{-1} v] + [\beta u + \gamma v, z]$. If all 2-generated subalgebras are free, then $[\beta u + \gamma v, \alpha\beta^{-1} v + z]$ cannot be a relator unless $\beta u + \gamma v, \alpha\beta^{-1} v + z$ are linearly dependent. This is never the case with non-trivial coefficients.

Definition 2.2.13. Let $K \in \widetilde{\mathcal{K}}_2$ be a finite strong extension of M, a *minimal decomposition* of K over M is a sequence of minimal self-sufficient extensions

$$M = M^0 \leqslant M^1 \leqslant \cdots \leqslant M^n = K. \tag{2.10}$$

such that the following two conditions are satisfied:

(1) If $d^K(K/M) = d$ for some $d \leq n$, then $M^i \supseteq M^{i-1}$ is transcendental for $i \leq d \leq n$,

(2) for all $i > d$, if M^i *is not* a minimal algebraic extension of M^{i-1}, then there is no divisor a of M^{i-1} in K.

Since the notion of self-sufficiency is transitive, by Lemma 2.1.16, it is always possible to find a minimal decomposition of K over M *for any* finite strong extension K of M. We first exhaust all transcendental steps and obtain M^d like in (1), so that $d^K(K/M^d) = \delta(K/M^d) = 0$.

Then (2) follows, by letting algebraic extensions take precedence over prælgebraic ones in the sequence.

With Proposition 2.3.12 of the next section follows that the number of prealgebraic steps in a minimal decomposition is an invariant of the elementary type of the extension.

Minimal decompositions commute with free amalgamation:

Lemma 2.2.14. *Let $M \geqslant M^0 \subseteq H^0$ be $\mathfrak{L}_p^2$-algebras. Then*

$$M^0 \leqslant M^1 \leqslant \cdots \leqslant M^n = M$$

is a minimal decomposition of M over M^0 if and only if

$$H^0 \leqslant M^1 \circledast_{M^0} H^0 \leqslant M^2 \circledast_{M^0} H^0 \leqslant \cdots \leqslant M^n \circledast_{M^0} H^0$$

is a minimal decomposition of $M \circledast_{M^0} H^0$ over H^0.

In any of the two cases above, each extension

$$M^i \circledast_{M^0} H^0 \supseteq M^{i-1} \circledast_{M^0} H^0$$

is exactly of the same kind of $M^i \supseteq M^{i-1}$, for all $1 \leq i \leq n$.

Proof. As for all i, Lemmas 2.2.6 and 2.2.7 imply

$$M^i \circledast_{M^0} H^0 \simeq M^i \circledast_{M^{i-1}} (M^{i-1} \circledast_{M^0} H^0) \geqslant M^{i-1} \circledast_{M^0} H^0,$$

both statements of the Lemma follow by considering M minimal over M^0.

Let then H denote the free amalgam $M \circledast_{M^0} H^0$. For any subspace K_1 of H_1 with $K_1 \supseteq H_1^0$, since $K_1 = H_1^0 + (K_1 \cap M_1)$, we have

$$H_1 \supsetneq K_1 \supsetneq H_1^0 \iff M_1 \supsetneq K_1 \cap M_1 \supsetneq M_1^0.$$

If now K denote $\langle K_1\rangle^H$ and $K' = \langle K_1 \cap M_1\rangle^M$, then by Corollary 2.2.8, $K \simeq K' \circledast_{M^0} H^0$. Hence by Lemma 2.2.7 and Proposition 2.2.6

$$K' \leqslant M \iff K \leqslant M \circledast_{K'} K = M \circledast_{K'} (K' \circledast_{M^0} H^0) = H.$$

This means $H \supseteq H^0$ is minimal exactly if $M \supseteq M^0$ is a minimal extension and proves the first statement, while the second, follows by equality (2.8) – here $\delta(K/H^0) = \delta(K'/M^0)$ – of Lemma 2.2.7.In particular for any $h \in H_1$ there is an m in M_1 such that $\delta(h/H^0) = \delta(m/M^0)$ and the lemma follows.

□

Before we prove the next step toward amalgamation, we need to analyse in detail the space of relators R^2 in the free amalgam. To do that, we have to find suitable bases for the subspaces of the vector-space amalgam, which ease the treatment of basic monomials. This is Lemma 4.2 in [Bau96].

Lemma 2.2.15. *Assume H is the free amalgam $M \circledast_N K$ and let E_1 be a finite subspace of $H_1 = M_1 \oplus_{N_1} K_1$.*

Assume there exists $n < \omega$ and subsets $\mathcal{U} = (u_i)_{i=1}^n$ and $\mathcal{V} = (v_i)_{i=1}^n$ of M_1 and K_1 respectively, such that $(u_i + v_i)$ is a $\mathbb{Z}_p$-basis of E_1 over $E_1 \cap M_1 + E_1 \cap K_1$.

Then the subset $\mathcal{U}\mathcal{V}$ of H_1 is linearly independent over $E_1 \cap M_1 + E_1 \cap K_1 + N_1$.

Proof. We follow an inductive argument over $n < \omega$. Assume the assertion holds for $1 \leq k \leq n-1$ and $\mathcal{U} = (u_i)$ and $\mathcal{V} = (v_i)$ for $1 \leq i \leq n$ are the sets mentioned in the statement. If we set $\hat{\mathcal{U}} = \{u_i \mid i < n\}$ and $\hat{\mathcal{V}} = \{v_i \mid i < n\}$, then $\hat{\mathcal{U}}\hat{\mathcal{V}}$ is linearly independent over $E_1 \cap M_1 + E_1 \cap K_1 + N_1$.

Set $\tilde{E}_1 := \langle E_1 \cap M_1, E_1 \cap K_1, \hat{\mathcal{U}}, \hat{\mathcal{V}}, u_n + v_n\rangle_{\mathbb{Z}_p}$ and notice that $u_n + v_n$ generates $\tilde{E}_1$ over $\tilde{E}_1 \cap M_1 + \tilde{E}_1 \cap K_1$ hence, by induction, $\{u_n, v_n\}$ is linearly independent over $\tilde{E}_1 \cap M_1 + \tilde{E}_1 \cap K_1 + N_1 = \langle E_1 \cap M_1, E_1 \cap K_1, N_1, \hat{\mathcal{U}}, \hat{\mathcal{V}}\rangle_{\mathbb{Z}_p}$ and the set $\mathcal{U}\mathcal{V}$ is independent over $E_1 \cap M_1 + E_1 \cap K_1 + N_1$.

The assertion is therefore to be proven in the case $n = 1$. Let then E_1 be generated by a sum $u + v$ over $E_1 \cap M_1 + E_1 \cap K_1$ for $u \in M_1$ and $v \in K_1$.

It follows $u+v$ is not in $E_1 \cap M_1 + E_1 \cap K_1 + N_1$. If now $su+tv \in E_1 \cap M_1 + E_1 \cap K_1 + N_1$ for some s and t in $\mathbb{Z}_p$ and say $s \neq 0$, we have then $(t-s)v \in K_1 \cap (E_1 + N_1) = N_1 + (E_1 \cap K_1)$ and thus $s(u+v) \in E_1 \cap M_1 + E_1 \cap K_1 + N_1$ which is a contradiction.

□

Remark 2.2.16. Let H be the free amlagam above. In the previous notation, for every finite $E_1 \subseteq H_1$, we can find a $\mathbb{Z}_p$-base of E_1 in the form

$$\mathcal{E}_N \mathcal{E}_M \mathcal{E}_K (u_i + v_i \mid u_i \in \mathcal{U}, v_i \in \mathcal{V})_{i=1,\ldots,n} \tag{2.11}$$

where $\mathcal{E}_N$ is a base of $E_1 \cap N_1$, and $\mathcal{E}_M$ and $\mathcal{E}_K$ complete $\mathcal{E}_N$ to a basis of $E_1 \cap M_1$ and $E_1 \cap K_1$ respectively.

A $\mathbb{Z}_p$-basis $\mathcal{H} = \mathcal{H}_M\mathcal{H}_N\mathcal{H}_K$ of H_1 is said *compatible* with the base (2.11), if $\mathcal{E}_N \subseteq \mathcal{H}_N$ and $\mathcal{H}_N$ is a $\mathbb{Z}_p$-basis of N_1; if $\mathcal{U}, \mathcal{E}_M \subseteq \mathcal{H}_M$ and $\mathcal{H}_M\mathcal{H}_N$ is a basis for M_1 and lastly if $\mathcal{V}, \mathcal{E}_K \subseteq \mathcal{H}_K$ and $\mathcal{H}_N\mathcal{H}_K$ is a basis for K_1.

This way of extending bases to H_1 leads to the following description of $R^2(E)$ for any given finite E.

Recall with Fact 1.4.7, Corollary 1.4.10 and Definitions 1.4.4 and 1.4.11, that for a base $\mathcal{H}$ of H_1, any set $\mathscr{B} = \mathscr{B}_{\leq 2}$ of basic monomials over $\mathcal{H}$ of weight ≤ 2, constitutes a basis of $L^2(H_1) = H_1 \oplus \bigwedge^2 H_1$. In particular chosen an order $(\mathcal{H}, <)$, the set $\mathscr{B}_2 = \{[b,c] \mid b > c \in \mathcal{H}\}$, is a basis of $\bigwedge^2 H_1$ and $\mathscr{B} = \{\mathcal{H} < \mathscr{B}_2\}$ a basis of $L^2(H_1)$. The following is borrowed from [Bau96, Lemma 4.3].

Lemma 2.2.17. *Let $E = \langle E_1 \rangle^H$ be a finite subalgebra of the free amalgam $H = M \circledast_N K$ for $M \supseteq N \subseteq K$ algebras in $\mathfrak{L}_p^2$.*

Let $\mathcal{E} = \mathcal{E}_N\mathcal{E}_M\mathcal{E}_K(u_i + v_i : i = 1, \dots, n)$ be a basis of E_1 as in (2.11).

We order a base $\mathcal{H} = \mathcal{H}_N > \mathcal{H}_M > \mathcal{H}_K$ of H_1 compatible with $\mathcal{E}$, in such a way that $\mathcal{E}_N > \mathcal{E}_M > \mathcal{U} > \mathcal{E}_K > \mathcal{V}$.

Then each element Φ of $R^2(E)$ has the form

$$\Phi_M + \Phi_K + \phi_u + \phi_v \tag{2.12}$$

where

Φ_M and Φ_K are linear combination of basic $\mathcal{H}$-commutators with support in $\mathcal{E}_N\mathcal{E}_M$ and $\mathcal{E}_N\mathcal{E}_K$ respectively.

ϕ_u is a linear combination of basic $\mathcal{H}$-commutators $[h, u_i]$ where h belongs to $\mathcal{E}_N$ and u_i is in $\mathcal{U}$

ϕ_v is obtained by replacing each instance of u_i in ϕ_u by the corresponding v_i from $\mathcal{V}$.

Finally there exists η in $\bigwedge^2 N_1$ such that $\Phi_M + \phi_u + \eta$ belongs to $R^2(M)$ and $\Phi_K + \phi_v - \eta$ is in $R^2(K)$.

Proof. Let Φ be an element of $R^2(E)$, which is by definition $R^2(H) \cap \bigwedge^2 E_1$. Now $R^2(H) = R^2(M) + R^2(K) \subseteq \bigwedge^2 M_1 + \bigwedge^2 K_1$, hence there exist ρ_M in $R^2(M)$ and $\rho_K \in R^2(K)$ such that $\Phi = \rho_M + \rho_K$.

Write ρ_M and ρ_K as linear combinations of basic $\mathcal{H}$-monomials with support respectively in $\mathcal{H}_N\mathcal{H}_M$ and $\mathcal{H}_N\mathcal{H}_K$, and call $\Phi_{\mathcal{H}}$ the resulting unique expression of basic $\mathcal{H}$-commutators which equals Φ after Corollary 1.4.10.

On the other hand, consider the linear order on $\mathcal{E}$ given by $\mathcal{E}_N > \mathcal{E}_M > \mathcal{E}_K > \{u_1 + v_1 > \dots > u_n + v_n\}$, where the first three fragments inherit the order of $\mathcal{H}$ above. Now write $\Phi \in \bigwedge^2 E_1$ as a linear combination $\Phi_{\mathcal{E}}$ of basic $\mathcal{E}$-commutators. By linearity, each monomial involving entries $u_i + v_i$ expands into a sum of basic monomials over $\mathcal{H}$: just transpose – and accordingly change the sign of – the entries which are in the wrong order. This means $\Phi_{\mathcal{E}}$ is actually equal to a linear combination Φ' of basic $\mathcal{H}$-monomials as well.

Now comparing expressions $\Phi_{\mathcal{H}} = \Phi = \Phi'$, by Corrollary 1.4.10, exactly the same monomials must appear in $\Phi_{\mathcal{H}}$ and Φ'.

It follows, that terms of the kind $[b, u_i + v_i]$ with $b \in \mathcal{E}_M\mathcal{E}_K$ or $b = u_j + v_j$ for any j, are not allowed in the expression $\Phi_{\mathcal{E}}$. Following the same argument, basic monomials $[m, k]$ with $m \in \mathcal{E}_M$ and $k \in \mathcal{E}_K$ are excluded from $\operatorname{supp}_{\mathcal{E}}(\Phi_{\mathcal{E}})$ as well.

We can conclude Φ consists of the sum $\Phi_M + \Phi_K + \phi_u + \phi_v$ described in the statement of the lemma.

To obtained η, consider equality $\Phi_M + \Phi_K + \phi_u + \phi_v = \Phi = \rho_M + \rho_K$ and set $\eta := \rho_M - \Phi_M - \phi_u = \Phi_K + \phi_v - \rho_K \in \bigwedge^2 M_1 \cap \bigwedge^2 K_1 = \bigwedge^2 N_1$.

□

With the above description of relators, we can prove the following lemma, which shows, the only obstruction for the free amalgam to inherit property $\Sigma^2(2)$ from its components are the divisor elements of the base.

If some algebra K extends N and $N \subseteq H$, a divisor (cfr. Remark 2.2.11) $a \in H_1$ of N is *realised* in K over N, if there exists an element b of K_1 and an isomorphism of $\langle N_1, a\rangle^H$ onto $\langle N_1, b\rangle^K$ which fixes N and maps a onto b.

If N is strong in both H and K, according to Remark 2.2.11, for some $x \in N_1$ and $\eta \in \bigwedge^2 N_1$, $[a, x] - \eta$ generates $R^2(N, a)$ over $R^2(N)$. In order to realise a in K, it is sufficient to find $b \in K_1$ with $[b, x] - \eta \in R^2(K)$.

Proposition 2.2.18. *Assume $M \overset{k}{\geqslant} N \leqslant K$ for $\widetilde{\mathcal{K}}_2$-algebras M, N and K, where K is a finite extension of N, and the integer k is not smaller than $\dim_{\mathbb{Z}_p}(K_1/N_1)$.*

Assume also that for any divisor a of N in K, a is not realised in M over N. Then $M \circledast_N K$ satisfies $\Sigma^2(2)$.

Proof. Let H denote $M \circledast_N K$, then by Lemma 2.2.7 we have $M \leqslant H \overset{k}{\geqslant} K$.

Let E_1 be a finite subspace of $M_1 \oplus_{H_1} K_1$ and choose a $\mathbb{Z}_p$-basis $\mathcal{E} = \mathcal{E}_N\mathcal{E}_M\mathcal{E}_K(u_i + v_i \mid i = 1, \ldots, n)$ of E_1 for suitable u_i's in M_1 and v_i's in K_1 as described in Remark 2.2.16. We have to show $\delta(E) \geq \min(2, \dim_{\mathbb{Z}_p}(E_1))$.

Applying submodularity (2.5) of δ, we find $\delta(E) \geq \delta(E/M) + \delta(E_1 \cap M_1)$. Since M is self-sufficient and satisfies $\Sigma^2(2)$, if $\dim_{\mathbb{Z}_p}(E_1 \cap M_1) \geq 2$ we are done. We might then assume $\dim_{\mathbb{Z}_p}(E_1 \cap M_1) < 2$.

If $E_1 \cap M_1 = \mathbf{0}$, then $\mathcal{E} = \mathcal{E}_K(u_i + v_i \mid i = 1, \ldots, n)$ and by Lemma 2.2.17 we have $R^2(E) = R^2(E_1 \cap K_1)$. It follows $\dim_{\mathbb{Z}_p}(E_1) = \dim_{\mathbb{Z}_p}(E_1 \cap K_1) + n$ and hence $\delta(E) = \delta(E_1 \cap K_1) + n$. This yields $\delta(E) \geq \min(2, \dim_{\mathbb{Z}_p}(E_1))$ since $\delta(\langle E_1 \cap K_1\rangle^K)$ does.

Assume $E_1 \cap M_1$ has dimension 1. If $E_1 \cap N_1 = \mathbf{0}$ then by Lemma 2.2.17 again, $R^2(E) = R^2(E_1 \cap K_1)$ because $\mathcal{E} = \{m\}\mathcal{E}_K(u_i + v_i \mid i = 1, \ldots, n)$ with $\{m\}$ as $\mathcal{E}_M$. We can conclude as above: this time $\delta(E) = \delta(E_1 \cap K_1) + n + 1$.

Assume now $E_1 \cap M_1 = E_1 \cap N_1 = \langle h\rangle$ has dimension 1. This implies $\mathcal{E} = \{h\}\mathcal{E}_K(u_i + v_i \mid i = 1, \ldots, n)$ and $E_1 = \langle E_1 \cap K_1, u_i + v_i \mid i = 1, \ldots, n\rangle$.

If $E_1 \cap K_1$ is $\langle h \rangle$ as well ($\mathcal{E}_K = \varnothing$), then $E_1 = \langle h, u_i + v_i \mid i = 1, \ldots, n \rangle$. If we assume that $R^2(E)$ is nontrivial then by Lemma 2.2.17 a nonzero element Φ of $R^2(E)$ is equal to a sum $\sum_{i=1}^n s_i[h, u_i + v_i]$ for some scalars s_i in $\mathbb{Z}_p$. Moreover for a suitable η in $\bigwedge^2 N_1$, $[h, \sum_{i=1}^n s_i u_i] + \eta$ lies in $R^2(M)$ and $[h, \sum_{i=1}^n s_i v_i] - \eta$ in $R^2(K)$.

If we now set $v := \sum_{i=1}^n s_i v_i \in K_1 \setminus N_1$, then we get $\delta(v/N) = 0$ and v is a divisor of K. Since N is at least 1-self-sufficient in M, if we set $u := -\sum_{i=1}^n s_i u_i$, then $[h, u] - \eta \in R^2(M)$ and u realises v in M over N. The hypotheses now imply that this sitution cannot occur and in this case $R^2(E) = \mathbf{0}$.

For the very last step we assume, as in the previous case, that $E_1 \cap M_1 = E_1 \cap H_1 = \langle h \rangle$, but this time $\dim_{\mathbb{Z}_p}(E_1 \cap K_1) \geq 2$ and thus $\delta(E_1 \cap K_1) \geq 2$.

Now by submodularity over K we obtain $\delta(E) \geq \delta(E/K) + \delta(E_1 \cap K_1)$ and $\delta(E/K) = \delta(\langle u_i + v_i \mid i = 1, \ldots, n \rangle / K) \geq 0$ since H is k-self-sufficient in M and by Lemma 2.2.15, $n \leq \min(\dim_{\mathbb{Z}_p}(M_1/N_1), \dim_{\mathbb{Z}_p}(K_1/N_1)) \leq k$.

As there is no more cases left, H has $\Sigma^2(2)$ and the proof is complete.

□

The following *asymmetric amalgamation* both proves amalgamation in $\widetilde{\mathcal{K}}_2$ and makes it possible to axiomatise the theory of the rich Fraïssé limit of $\mathcal{K}_2$.

Lemma 2.2.19 (Asymmetric Amalgam). *Let M, N and K be algebras of $\widetilde{\mathcal{K}}_2$ such that K is a finite self-sufficient extension of N, and N is $n + \dim_{\mathbb{Z}_p}(K_1/N_1)$-self-sufficiently embedded in M, for some $n < \omega$.*

Then there exists H in $\widetilde{\mathcal{K}}_2$, which amalgamates M and K over N as in Definition 2.2.1, under which embeddings, M is strong and K is n-strong in H.

Proof. Fix an integer n and assume M, N and K as above. We prove the statement by induction on $l = \dim_{\mathbb{Z}_p}(K_1/N_1)$. So let $\widetilde{N} = \langle \widetilde{N}_1 \rangle^K$ be a self-sufficient subalgebra of K such that $K \supsetneq \widetilde{N}$ is a minimal strong extension. [7] Denote by $\widetilde{\nu}$ the strong embedding of N into $\widetilde{N}$, by ν' the strong embedding of $\widetilde{N}$ into K and by ϕ the embedding of N into M.

If $\widetilde{l} = \dim_{\mathbb{Z}_p}(\widetilde{N}_1/N_1)$ and $l' = \dim_{\mathbb{Z}_p}(K/\widetilde{N}_1)$, then by the inductive hypothesis there exists an algebra $\widetilde{M}$ in $\widetilde{\mathcal{K}}_2$ which amalgamates M and $\widetilde{N}$ over N by virtue of a strong embedding $\widetilde{\mu}$ of M into $\widetilde{M}$ and of a $n + l'$-strong embedding ψ' of $\widetilde{N}$ into $\widetilde{M}$ such that $\phi\widetilde{\mu} = \widetilde{\nu}\psi'$.

[7] consider the last step of a minimal decomposition of K over N.

$$
\begin{tikzcd}
 & & H & \\
 & \widetilde{M} \arrow[ur, dash, "\mu'"] & & K \arrow[ul, dash, dashed, "\psi"'] \\
 & & \widetilde{N} \arrow[ul, dash, dashed, "\psi'"] \arrow[ur, dash, "\nu'"'] & \\
M \arrow[uur, dash, "\widetilde{\mu}"] \arrow[dr, dash, dashed, "\phi"] & & & \\
 & N \arrow[uur, dash, "\widetilde{\nu}"'] & &
\end{tikzcd}
\tag{2.13}
$$

Now we distinguish two cases: in the first one, K is an algebraic extension of $\widetilde{N}$ with $K = \langle \widetilde{N}_1, a \rangle$ for some a in K_1 and we assume that K is realised in $\widetilde{M}$ over $\widetilde{N}$ by means of an element m of $\widetilde{M}_1$. We set in this case $H = \widetilde{M}$ and let ψ denote the $\mathfrak{L}_p^2$-Lie isomorphism which fixes $\widetilde{N}$ and maps a onto m. Then clearly H has $\Sigma^2(2)$. Now since $\widetilde{N}$ is $n+1$-selfsufficient in H and $\delta(m/\widetilde{N}) = 0$, it follows that $\psi(K)$ is n-self-sufficient in H. This is clear since, for any finite subspace E_1 of H_1 with $\dim_{\mathbb{Z}_p}(E_1/\psi(K_1)) \leq n$ one has $\delta(E/\psi(K)) = \delta(E/\widetilde{N}_1, m) = \delta(E_1, m/\widetilde{N}) \geq 0$.

In the second case we consider algebraic extensions which *are not* realised in H, as well as free or pre-algebraic extensions $K/\widetilde{N}$.

By Proposition 2.2.18 the free amalgam $\widetilde{M} \circledast_{\widetilde{N}} K =: H$ satisfies property $\Sigma^2(2)$. Denote with ψ the canonical embedding of K into H, as ψ' is $n+l'$-strong, Lemma 2.2.7 implies that ψ is $n+l'$-strong as well, and in particular n-strong.

In both of the cases considered above, denote with μ' the strong embedding of $\widetilde{M}$ into H and put $\mu := \widetilde{\mu}\mu'$ and $\nu := \widetilde{\nu}\nu'$. Then μ is a strong embedding, and $\phi\mu = \nu\psi$ as required by Definition 2.2.1.

□

Corollary 2.2.20. *The countable class $\mathcal{K}_2$ has the properties (HP), (JEP) and (AP) defined in Section 1.2, with respect to strong $\mathfrak{L}_p^2$-embeddings.*

Proof. As $\Sigma^2(2)$ is expressed by universal sentences, then in particular $\mathfrak{L}_p^2$-subalgebras of an object A of $\mathcal{K}_2$ still satisfy it. Hence we have (HP).

That $\mathcal{K}_2$ satisfies (AP) is Corollary 2.2.19, with M, N and K finite and strong embeddings on both sides.

Moreover, since the trivial algebra $\mathbf{0}$ is self-sufficient in every structure of $\mathcal{K}_2$, if we apply Corollary 2.2.19 to pairs of algebras (with $N = \mathbf{0}$) we obtain the joint embedding property (JEP).

□

Now Fact 1.2.2 applies to the countable class $\mathcal{K}_2$ with respect to strong $\mathfrak{L}_p^2$-embeddings. The Fraïssé limit $\mathbb{K}$ of $(\mathcal{K}_2, \leqslant)$ obtained in this way, is a countable $\mathcal{K}_2$-*rich* algebra of $\widetilde{\mathcal{K}}_2$. This means by definition

- $\text{âge}(\mathbb{K}) = \mathcal{K}_2$
- for any finite strong $\mathfrak{L}_p^2$-extension A of B in $\mathcal{K}_2$, if $\beta\colon B \to \mathbb{K}$ is a self-sufficient embedding, there exists a strong $\mathfrak{L}_p^2$-embedding α of A into $\mathbb{K}$ with $\alpha\restriction_B = \beta$.

2.3 A first order Theory for the Fraïssé Limit

In this last section, we axiomatise the $\mathscr{L}^2$-theory of the Fraïssé limit $\mathbb{K}$ of $(\mathcal{K}_2, \leqslant)$. We prove it is ω-stable and calculate its Morley rank.

Nilpotency of class 2 can be expressed universally in $\mathscr{L}^2$, in terms of simple commutators, by requiring $[x, y, z] = [[x, y], z] = \mathbf{0}$ for all x, y, z. Altough the language $\mathscr{L}^2$ can naturally express the grading on each $\mathfrak{L}_p^2$-algebra, by means of $M = P_1(M) + P_2(M)$, $P_1(M) \cap P_2(M) = \mathbf{0}$ and $(\forall xy)P_2([x, y])$, in general there is no first-order bound to the length of homogeneous sums of weight 2, which could express $\langle P_1(M)\rangle = M$.

In the axiom system chosen below for $\mathbb{K}$, this is sorted out in the strongest way possible: we require each weight 2 element to be the Lie bracket of exactly two elements (from P_1). This feature may be compared with a corollary to Zilber's *Indecomposability Theorem*: in any group G of finite Morley rank, there exists an integer n such that any $g \in G'$ is the product of n commutators $[x_i, y_i]$.

As a consequence, it will be true that elementary $\mathscr{L}^2$-extensions are $\mathfrak{L}_p^2$-extensions.

If M is an $\mathscr{L}^2$-structure, we define the theory T^2 by means of the following denumerable first order schema of $\mathscr{L}^2$-axioms, expressed in terms of M:

($\Sigma^2(1)$) M is a graded nil-2 Lie algebra over the field $\mathbb{Z}_p$. This corresponds to write properties 1. and 2. of Definition 1.4.8 in $\mathscr{L}^2$ as described above.

($\Sigma^2(2)$) For any finite subspace H_1 of M_1 $\delta(H) \geq \min(\dim_{\mathbb{Z}_p}(H_1), 2)$.

($\Sigma^2(3)$) for any finite strong extension $A \supseteq B$ of $\mathcal{K}_2$-algebras and any $n < \omega$, if B is $(\dim_{\mathbb{Z}_p}(A_1/B_1) + n)$-selfsufficient in M, then there exists an isomorphic copy of A in M over B, which is n-self-sufficient.

($\Sigma^2(4)$) for all $y \in M$ with $P_2(y)$ and all $\mathbf{0} \neq z \in M_1$ there is $x \in M_1$ such that $y = [z, x]$

We can first observe that axioms $\Sigma^2(2)$ and $\Sigma^3(3)$ imply that a model of T^2 cannot be finite.

Theorem 2.3.1. *An $\mathscr{L}_2$-structure K is a rich algebra of $\widetilde{\mathcal{K}}_2$ if and only if K is an ω-saturated model of T^2.*

Proof. We start proving that a rich algebra K of $\widetilde{\mathcal{K}}_2$ is also a model of T^2 which is henceforth consistent: the Fraïssé limit $\mathbb{K}$ of $(\mathcal{K}_2, \leqslant)$ exhibits a countable model. The second part of the proof shows that an ω-saturated model of T^2 is a rich $\widetilde{\mathcal{K}}_2$-algebra, now since rich structures are $\mathscr{L}^2_{\infty,\omega}$-equivalent, because of Fact 1.2.2 ($\mathfrak{L}^2_p$-embeddings are in particular $\mathscr{L}^2$-embeddings), it follows that rich $\widetilde{\mathcal{K}}_2$-structures are ω-saturated.

So let first K be such a rich algebra in $\widetilde{\mathcal{K}}_2$, then axioms $\Sigma^2(1)$ and $\Sigma^2(2)$ are satisfied automatically.

To prove $\Sigma^2(3)$, assume A is a finite strong extension in $\mathcal{K}_2$ of a finite subalgebra B of K and B is $(\dim_{\mathbb{Z}_p}(A_1/B_1)+n)$-selfsufficient in K. Take a finite strong subalgebra $\widetilde{B}$ of K containing B (the selfsufficient closure of B for instance). We use the asymmetric amalgamation Lemma 2.2.19 to obtain a strong extension $\widetilde{A}$ of $\widetilde{B}$ in $\mathcal{K}_2$, such that A is n-self-sufficient in $\widetilde{A}$.

Now since K is rich, $\widetilde{A}$ strongly embeds into K over $\widetilde{B}$. As a consequence of transitivity (Lemma 2.1.11), A embeds into K n-selfsufficiently over B.

To prove $\Sigma^2(4)$, pick an element $w \in K_2$ and $m \in K_1$. Since $K = \langle K_1 \rangle$, there exists a finite subspace B_1 of K_1 with $m \in B_1$ and $w \in B_2$. We may clearly assume that B is self-sufficient in K.

If there exists c in B_1 such that $[c, m] = w$ we are done, if not, then we can apply Remark 2.2.12, and find a minimal algebraic strong extension A of B, such that A is in $\mathcal{K}_2$ and $A_1 = \langle B_1, a \rangle_{\mathbb{Z}_p}$ where $[a, m] = w$. Now since $B \leqslant K$ and K is rich, a is realised in K over B. In particular there is a' in K_1 with $[a', m] = w$ as desired.

For the reverse implication suppose M is an ω-saturated model of T^2, then by $\Sigma^2(4)$ the $\mathscr{L}^2$-structure M is in particular an object of $\mathfrak{L}^2_p$.

Now let $A \supseteq B$ be a finite strong extension of $\mathcal{K}_2$-algebras, where B is a finite strong $\mathfrak{L}^2_p$-subalgebra of M. We may assume, without loss of generality, that A is a minimal extension of B; otherwise we decompose it in a chain of minimal strong sections like (2.10) and strongly embed each subalgebra stepwise in M, over the predecessor.

Assume first $A \supseteq B$ is a free extension and B is a finite strong substructure in M, by Proposition 2.2.9 and Lemma 2.1.16 we are done if we find $a \in M_1$ which is $\mathscr{cl}_d$-independent of B_1, as $d(a/B_1) = 1$ implies that $\langle B_1, a \rangle^M$ is strong in M

If we can prove that $d^M(M_1)$ is infinite, the desired condition will follow. To do so, denote by $L^{2,n}(\bar{x})$ the $\mathscr{L}^2$-formula, which describes the finite free nil-2 Lie algebra in the following way: for any n-tuple $\bar{a}$ in M_1, $M \models L^{2,n}(\bar{a})$ means $\langle a_1, \ldots, a_n \rangle^M \simeq L^2(\bar{a})$.

We show, with an inductive argument, that we can *strongly* embed $L^{2,n}(\bar{x})$ in M for any $n < \omega$. Axiom $\Sigma^2(2)$ ensure that for any independent pair m_1, m_2 of M_1, $\langle m_1, m_2 \rangle^M$ is a selfsufficient subalgebra of M isomorphic to the free nilpotent algebra $L^2(m_1, m_2)$; this will be our inductive base.

Assume now $M \models L^{2,n}(\bar{b})$ and $\langle \bar{b} \rangle^M$ is strong in M. Consider the collection $\Phi^{n+1}(x, \bar{b})$ of all formulae $\phi^{n+1}_k(x, \bar{b})$ for $k < \omega$, where a $\mathscr{L}^2$-structure L satisfies $\phi^l_k(\bar{y})$ in $\bar{m}$ exactly if $\bar{m} \subseteq L_1$, $L \models L^{2,l}(\bar{m})$ and $\langle \bar{m} \rangle^L$ is k-strong in L.

Now a finite portion of Φ^{n+1} is implied by a single formula $\phi^{n+1}_k(x, \bar{b})$ with a sufficiently large k. Now since $\langle \bar{b} \rangle^M$ is strong, we have $M \models \phi^n_{k+1}(\bar{b})$ and hence by $\Sigma^2(3)$ there exists a in M_1 such that $M \models \phi^{n+1}_k(a, \bar{b})$.

We showed $\Phi^{n+1}(x,\bar{b})$ is consistent with $T^2_{\bar{b}}$ and hence realized in M_1 by ω-saturation for some $m \in M_1$. It follows $\langle m,\bar{b}\rangle^M$ is selfsufficient and hence $d^M(m,\bar{b}) = \delta(m,\bar{b}) = n+1$. By induction, M has infinite d-dimension.

If $A \supseteq B$ is a minimal strong extension with $\delta(A/B) = 0$ (hence algebraic or prealgebraic). Since B is strong in M, (anyone among) axioms $\Sigma^2(3)$ ensure the existence of an isomorphic copy A' of A in M over B. Note that algebraic extension may be sorted out with $\Sigma^2(4)$ as well.

Because of $\delta(A'/B) = 0$, we have that A' is self-sufficient in M as well. This proves that M is a rich Lie algebra in $\mathcal{K}_2$.

□

The proof of the theorem also shows that if M is a κ-saturated model of T^2, then its $c\ell_d$ dimension $d(M)$ is not smaller than κ.

Note that the Fraïssé limit $\mathbb{K}$ of $\mathcal{K}_2$ in the last section, is "the" countable saturated model of T^2.

Lemma 2.3.2. *Elementary $\mathfrak{L}^2_p$-embedding are strong. In particular, elementary $\mathscr{L}^2$-extension of models of T^2 are strong $\mathfrak{L}^2_p$-extensions.*

Proof. Let M be an elementary $\mathfrak{L}^2_p$-subalgebra of N.

If M is not strong in N, $\delta(A/M) < 0$ for some finite subspace A_1 of N_1. By Remark 2.1.10 there is a finite strong C_1 in M_1 such that $\delta(A/M) = \delta(A/C)$. But then, since now $\delta(A/C)$ is expressible through a formula over C, for some finite subspace A'_1 of M_1 we have $\delta(A'_1/C_1)$ contradicting self-sufficiency of C in M.

□

Proposition 2.3.3. *Assume M and M' are two models of T^2. Let $\bar{a}$ and $\bar{a}'$ be tuples of M_1 and M'_1 respectively.*

Then $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{a}')$ if and only if the selfsufficient closure $\langle ssc_2(\bar{a})\rangle^M$ is $\mathfrak{L}^2_p$-isomorphic to $\langle ssc_2(\bar{a}')\rangle^{M'}$ via a Lie isomorphism mapping $\bar{a}$ onto $\bar{a}'$.

Proof. If we assume $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{a}')$, then we have $d^M(\bar{a}) = d^{M'}(\bar{a}')$ and we can find a finite subspace A_1 of M_1 containing $\bar{a}$ and isomorphic to $ssc^{M'}(\bar{a}')$. This yields $A_1 = ssc^A(\bar{a})$. Now since $\delta(A) = d^M(\bar{a}') = d^M(\bar{a}) \leq d^M(A_1)$, A is strong in M, it follows $A_1 = ssc^M(\bar{a})$ by Lemma 2.1.15.

For the other direction we may assume that M and M' are ω-saturated, since by Lemma 2.3.2 the self-sufficient closure of a subspace of M_1 will remain the same if computed in any elementary (saturated) extension of M.

Assume tuples $\bar{b} \subseteq M_1$ and $\bar{b}' \subseteq M'_1$ generate isomorphic strong subalgebras in M and M' respectively, we show that $\bar{b}$ and $\bar{b}'$ can be matched up by an Ehrenfeucht-Fraïssé game of lenght ω. This implies that an isomorphism between $\langle\bar{b}\rangle^M$ and $\langle\bar{b}'\rangle^{M'}$ preserves $\mathscr{L}_{\infty,\omega}$-formulas, hence $\mathrm{tp}(\bar{b}) = \mathrm{tp}(\bar{b}')$.

Assume one player chooses an element m of M – say – outside $\langle\bar{b}\rangle^M$. Then the other player first adds a linear independent tuple $\bar{c}$ over $\bar{b}$ such that $m \in \langle\bar{b},\bar{c}\rangle^M$ and such that

$\langle \bar{b}, \bar{c} \rangle \leqslant M_1$. Since $\langle \bar{b}' \rangle$ is strong embeddable into $\langle \bar{b}, \bar{c} \rangle$ and M' is a rich $\widetilde{\mathcal{K}}_2$-structure, one can respond with a tuple $\bar{c}'$ of M_1' with $\langle \bar{b}, \bar{c} \rangle^M \simeq \langle \bar{b}', \bar{c}' \rangle^M$ and $\langle \bar{b}', \bar{c}' \rangle \leqslant M_1'$. We can play ω rounds in this way, back-and-forth between M and M'.

□

Remark. Proposition 2.3.3 allows a somewhat opposite statement of Lemma 2.3.2: any self-sufficient extension of models of T^2 is elementary.

Since **0** is self-sufficient in every model, by the lemma above we obtain that the theory T^2 is complete and in general any two algebras $H \leqslant M$ and $H' \leqslant M'$ which are self-sufficient in models M and M' of T^2 do have the same elementary type if and only if they are isomorphic.

For the rest of the chapter, we assume a large saturated model $\mathbb{M}$ has been fixed, as monster model of T^2. By the above remarks, any model M of T^2 is a self-sufficient $\mathfrak{L}^2_p$-subalgebra of $\mathbb{M}$ with $|M| < \mathbb{M}$ and in particular, by Lemma 2.1.15 $d^M = d^{\mathbb{M}}$ on M_1 for any model M. Since the theory will be proved to be ω-stable, for the most purposes the countable saturated model $\mathbb{K}$ will be enough.

As an immediate corollary of the previous proposition and Lemma 2.1.16 we have

Remark 2.3.4. For any strong H in $\mathbb{M}$, any a, a' in $\mathbb{M}_1$ are $c\ell_d$-independent of H – that is $d(a/H_1) = d(a'/H_1) = 1$ – exactly if $\mathrm{tp}(a/H) = \mathrm{tp}(a'/H)$.

On the other hand by Proposition 2.1.16 and 2.3.3 we obtain

Corollary 2.3.5. *Let B be a finite strong subalgebra of a model M of T^2.*
Assume $\bar{a}$ is a tuple in M_1 such that $d(\bar{a}/B_1) = 0$. Let the $\mathscr{L}^2_B$-formula $\Delta(\bar{x}, \bar{y})$ describe the quantifier-free diagram of $ssc(B, \bar{a})$ in such a way that for any tuple $\bar{c}$ of M_1, for M to satisfy $\Delta(\bar{a}, \bar{c})$ means that $\langle B_1, \bar{a}, \bar{c} \rangle^M \simeq ssc(B, \bar{a})$. Then the formula $\exists \bar{y} \Delta(\bar{x}, \bar{y})$ isolates $\mathrm{tp}(\bar{a}/B_1)$.

We will now prove that our theory T^2 is totally transcendental. The outline of the proof below is borrowed from Wagner's [Wag94].

Proposition 2.3.6. *T^2 is ω-stable.*

Proof. Proposition 2.3.6 Since $\mathbb{M} = \langle \mathbb{M}_1 \rangle^{\mathbb{M}}$, it is sufficient to count types $\mathrm{tp}(\bar{m}/H)$ for tuples $\bar{m}$ in $\mathbb{M}_1$ and countable sets $H \subseteq \mathbb{M}$ (cfr. 2.3.10). Moreover without loss of generality we might assume that $H = \langle H_1 \rangle^{\mathbb{M}}$ is a self-sufficient subalgebra of (or a countable model in) $\mathbb{M}$.

The type of $\bar{m}$ over H is fully determined by the quantifier-free type of $\langle ssc(H_1, \bar{m}) \rangle^{\mathbb{M}}$. By Lemma 2.1.17 we have $ssc(H_1, \bar{m}) = \langle H_1, \bar{a} \rangle_{\mathbb{Z}_p}$ for a finite tuple $\bar{a}$ of $\mathbb{M}_1$, linearly independent over H_1. Moreover – still by Lemma 2.1.17 – we can find a finite subalgebra A and B with $B \leqslant H$ and $A_1 = \langle B_1, \bar{a} \rangle_{\mathbb{Z}_p}$ such that $\delta(A/B) = \delta(A/H)$ and $A_1 \cap H_1 = B_1$.

By Lemma 2.2.4 H and A are in free composition over B, that is

$$\langle ssc(H_1, \bar{m}) \rangle^{\mathbb{M}} = H + A \simeq H \circledast_B A.$$

Since the isomorphism type of the free amalgam is fully determined by its components, the type of $H + A$ is determined by $\mathrm{tp}(A_1/B)$ and by $\mathrm{tp}(B_1/H)$ – that is by the choice of B_1 into H_1.

Since we have a countable saturated model, namely the Fraïssé limit $\mathbb{K}$ of $\mathcal{K}_2$, the theory T^2 is small, this gives only countably many choices for $\mathrm{tp}(A_1/B)$. Altogether we have $\aleph_0 \cdot |H_1|^{<\aleph_0} = \aleph_0$ possibilities for $\mathrm{tp}(\bar{a}/H)$ and in particular for $\mathrm{tp}(\bar{m}/H)$.

$\square$

Remark 2.3.7. Any ω-saturated model M of T^2, satisfies a stronger version of richness over $\widetilde{\mathcal{K}}_2$. That is for *any* self-sufficient $\mathfrak{L}^2_p$-subalgebra N of M, if H is a finite strong extension of N in $\widetilde{\mathcal{K}}_2$, then M embeds H self-sufficiently over N.

Proof. Split H_1/N_1 into two strong sections H_1/K_1 and K_1/N_1 (cfr. Definition 2.2.13), such that $\delta(H/K) = 0$ and $d(H/N) = d(K/N) = \dim_{\mathbb{Z}_p}(K_1/N_1)$.

Now by saturation of M, iterating Remark 2.3.4 above, we first find a strong $\mathfrak{L}^2_p$-subalgebra $\widetilde{K}$ of M with $N \subseteq \widetilde{K}$ and $\widetilde{K} \simeq_N K$.

Secondly we consider the strong embedding $\widetilde{K} \hookrightarrow H$ and find – by Proposition 2.1.17 and the arguments of the previous Proposition – a finite $K^{\circ} \leqslant \widetilde{K}$ such that $H \simeq H^{\circ} \circledast_{K^{\circ}} \widetilde{K}$ for a suitable finite $H^{\circ} \subseteq H$ such that $H = K + H^{\circ}$.

With richness of M, find a strong embedding α of H° into M over K°. Now since $\delta(\alpha(H^{\circ})/\widetilde{K}) = \delta(H^{\circ}/\widetilde{K}) = \delta(H^{\circ}/K^{\circ}) = 0$, we obtained the desired strong embedding of H into M as $\langle \widetilde{K} + \alpha(H^{\circ})\rangle^M$.

$\square$

The next paragraphs are devoted to describe the algebraic closure of sets of $\mathbb{M}_1$.

First observe that axioms $\Sigma^2(2)$ imply that $\mathrm{Aut}(\mathbb{M})$ is 2-transitive on $\mathbb{M}_1$ as a group of $\mathbb{Z}_p$-linear automorphisms, that is to say, transitive on the set of linearly independent ordered pairs from $\mathbb{M}$. In particular $\mathrm{acl}(\mathbf{0}) = \mathbf{0}$, and $\mathrm{acl}(a, b) = \langle a, b\rangle^{\mathbb{M}}$ for any pair of elements $a, b \in \mathbb{M}_1$.

Now take a finite subspace C_1 of $\mathbb{M}_1$, since by saturation $d(\mathbb{M}_1/C)$ is infinite, Remark 2.3.4 implies that for any , $\mathrm{acl}(C_1) \cap \mathbb{M}_1$ is contained in $\mathcal{cl}_d(C_1)$.

It is also straightforward to see that $A = ssc(C)$ is contained in the algebraic closure of C: if A has infinitely many conjugates in $\mathbb{M}$ over C, then we can find a strong copy A' of A such that $C \subseteq A \cap A' \subsetneq A$ but his contradicts minimality of self-sufficient closure. With Lemma 2.1.14, we may also conclude that $\mathrm{acl}(C_1) \cap \mathbb{M}_1$ is self-sufficient. One has then

$$ssc(C_1) \leqslant \mathrm{acl}(C_1) \cap \mathbb{M}_1 \leqslant \mathcal{cl}_d(C_1). \tag{2.14}$$

As opposed to amalgamation constructions in pure relational languages (see [Wag94]), we will see below that the self-sufficient closure does not equal algebraic closure. On the other hand in our theory T^2 the algebraic closure does not coincide with the geometric closure $\mathcal{cl}_d$ either. This actually happens in collapsed structures.

We also have
$$\mathrm{acl}(C_1) = \langle \mathrm{acl}(C_1) \cap \mathbb{M}_1 \rangle^{\mathbb{M}} \tag{2.15}$$
for if C_1 is not trivial, for any element $m = m_1 + m_2$ of $\mathrm{acl}(C_1)$, property $\Sigma^2(4)$ implies $m = m_1 + [h, x]$ for some $h \in C_1$ and some x in $\mathbb{M}_1$.

Now $m_1 \in \mathrm{acl}(C_1) \cap \mathbb{M}_1$ and x is algebraic over m_1, h_1, by axiom $\Sigma^2(2)$ (cfr. Remark 2.3.10).

We can actually fully characterise $\mathrm{acl}(C_1)$ for a given $C_1 \subseteq \mathbb{M}_1$, in terms of the divisor elements defined in Remark 2.2.11.

Call a self-sufficient subalgebra C of $\mathbb{M}$ *divisibly closed* if whenever $\delta(a/C) = 0$ for $a \in \mathbb{M}_1$, then $a \in C_1$.

By the remarks above, $\mathrm{acl}(C_1)$ is divisibly closed. Moreover if U and V are divisibly closed $\mathfrak{L}_p^2$-algebras then $W = \langle U_1 \cap V_1 \rangle^{\mathbb{M}}$ is also divisibly closed, for if $\delta(x/U_1 \cap V_1) = 0$ then $\delta(x/U) = \delta(x/V) = 0$ and $x \in U_1 \cap V_1$.

Since meet-closed classes give rise to closure operators, we let $\mathscr{D}_C$ denote the collection of all subspaces H_1 containing C_1, which generate divisibly closed self-sufficient algebras in $\mathbb{M}$, then set
$$\mathrm{div}(C_1) = \bigcap \mathscr{D}_C$$
and consistently to our terminology $\mathrm{div}(C) = \langle \mathrm{div}(C_1) \rangle^{\mathbb{M}}$.

Lemma 2.3.8. *For any subspace C_1 of $\mathbb{M}_1$ we have*
$$\mathrm{acl}(C_1) = \mathrm{div}(ssc(C))$$

Proof. Since $ssc(C_1) \subseteq \mathrm{acl}(C_1)$, we may actually assume C to be self-sufficient and finite. As $\mathrm{acl}(C_1) = \langle \mathrm{acl}(C_1) \cap \mathbb{M}_1 \rangle^{\mathbb{M}}$ is divisibly closed, it is enough to show that $\mathrm{div}(C_1)$ contains $\mathrm{acl}(C_1) \cap \mathbb{M}_1$.

Assume an element a of $\mathbb{M}_1$ is in $\mathrm{acl}(C_1)$, let A_1 be $ssc^{\mathbb{M}}(C_1, a)$ and B_1 denote $ssc(C_1, a) \cap \mathrm{div}(C_1)$. Suppose by contradiction A is not included in $\mathrm{div}(C_1)$, then by (2.14) we have a non-trivial finite strong extension A of B such that $d(A/B) = \delta(A/B) = 0$.

Take distinct B-isomorphic copies $A = A^1$, A^2, $\ldots$, A^n of A for $n < \omega$; set $\circledast_B^0 A = B$ and $\circledast_B^1 A = A^1$. For all $1 \leq n < \omega$, also define inductively $\circledast_B^n A = (\circledast_B^{n-1} A) \circledast_B A^n$.

Then $\circledast_B^n A$ is in $\mathcal{K}_2$ for all n. This follows by Lemma 2.2.18: since B is divisibly closed in A^n, there is no divisor of B in A_1 to prevent the free amalgam of $\circledast_B^{n-1} A$ and A^n over B from satisfying property $\Sigma^2(2)$.

Since $\mathbb{M}$ is $\mathcal{K}_2$-rich, we can strongly embed $\circledast_B^n A$ into $\mathbb{M}$ over $\circledast_B^{n-1} A$ for all $n < \omega$. We obtain thus arbitrarily many distinct self-sufficient copies A^i of A over B and hence infinitely many C-conjugates of A in $\mathbb{M}$ against algebraicity over C.

□

By Definition 2.2.13 now follows

Remark 2.3.9. Let A finitely extend B in $\mathbb{M}$. Assume

$$B = B^0 \leqslant B^1 \leqslant \ldots \leqslant B^n = A$$

is a minimal decomposition of A over B. Then $\mathrm{acl}(B_1) \cap A_1 = B_1^k$, for some $1 \leq k \leq n$ such that $B^{i+1} \supseteq B^i$ is a minimal algebraic extension for all $i = 1, \ldots, k$ and k is maximal with respect to this property.

2.3.1 Rank Computations

Recall from Section 1.3, that between tuples $\bar{a}$ and small sets $\mathcal{A}, \mathcal{B}$ of $\mathbb{M}$, forking independence relation $\bar{a} \mathop{\smile\!\!\!\!\!\!\;|}^{f}_{\mathcal{B}} \mathcal{A}$ holds whenever $\mathrm{MR}(\bar{a}/\mathcal{B}) = \mathrm{MR}(\bar{a}/\mathcal{A}\mathcal{B})$.

We will use the following notation and facts in the sequel.

Remark 2.3.10. For a fixed non-trivial element m of $\mathbb{M}_1$ we define:

$$\begin{aligned} \vartheta_m \colon \mathbb{M}_1 \times \mathbb{M}_1 &\longrightarrow \mathbb{M} \\ (a_1, a_2) &\longmapsto a_1 + [m, a_2] \end{aligned} \tag{2.16}$$

for all a_1, a_2 in $\mathbb{M}_1$. Note that θ_m is a $\mathbb{Z}_p$-linear (non-bilinear) morphism. We have

1. For any fixed $m \neq \mathbf{0}$ of $\mathbb{M}_1$ the map θ_m defined in above is surjective and its fibres are all isomorphic to $\mathbb{Z}_p$.
2. For any tuple $\bar{a}$ of $\mathbb{M}$ and small sets $\mathcal{A}, \mathcal{B}$ there exists a tuple $\bar{a}'$ of $\mathbb{M}_1$ and *subspaces* A_1, B_1 of $\mathbb{M}_1$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\!\;|}^{f}_{\mathcal{B}} \mathcal{A}$ iff $\bar{a}' \mathop{\smile\!\!\!\!\!\!\;|}^{f}_{B_1} A_1$.
3. For any tuple $\bar{a} \subseteq \mathbb{M}$ and set $\mathcal{A} \subseteq \mathbb{M}$ there exists an $\mathfrak{L}_p^2$-subalgebra A of $\mathbb{M}$ interalgebraic with $\mathcal{A}$ such that $\mathrm{MR}(\bar{a}/\mathcal{A}) = \mathrm{MR}(\bar{a}/A) = \mathrm{MR}(\bar{a}/A_1)$.

Proof. Statement 1. follows by Axiom $\Sigma^2(4)$ for the surjectivity, while by $\Sigma^2(2)$ its fibres have all size exactly p: as pointed out in Section 2.2, $[m, u] = [m, v]$ implies necessarily $u - v$ linearly depends of m.

For 2. choose an element m in $\mathbb{M}_1$ with $m \mathop{\smile\!\!\!\!\!\!\;|}^{f}_{\mathcal{B}} \mathcal{A}, \bar{a}$. The properties of the definable map ϑ_m above, allow us to find subspaces $B_1 \subseteq A_1 \subseteq \mathbb{M}_1$ with $m \in B_1$ and a tuple $\bar{a}' \subseteq \mathbb{M}_1$, such that $B_1 \subseteq \mathrm{acl}(m, \mathcal{B})$, $\mathcal{B} \subseteq \mathrm{dcl}(B_1)$, $A_1 \subseteq \mathrm{acl}(m, \mathcal{B}, \mathcal{A})$, $\mathcal{A} \subseteq \mathrm{dcl}(A)$, $\bar{a}' \in \mathrm{acl}(m, \bar{a})$ and $\bar{a} \subseteq \mathrm{dcl}(m, \bar{a}')$.

Some forking calculus (Remark 1.3.6) now yields

$$\bar{a} \mathop{\smile\!\!\!\!\!\!\;|}^{f}_{\mathcal{B}} \mathcal{A} \iff \bar{a} \mathop{\smile\!\!\!\!\!\!\;|}^{f}_{m,\mathcal{B}} \mathcal{A} \iff \bar{a}' \mathop{\smile\!\!\!\!\!\!\;|}^{f}_{B} A$$

and hence the desired equivalence.

3. is proven by similar same arguments.

$\square$

With a fine description of types in T_2 *à la John B. Goode* ([Goo90]) it will be possible to calculate Morley rank of $\mathbb{M}$.

Theorem 2.3.11. *T^2 has Morley Rank $\omega \cdot 2$ and Morley degree 1.*

The crucial step in the proof relies in the following proposition. With $S(B)$ for $B \subseteq \mathbb{M}$ we denote the union of all $S_n(B)$ as n ranges in ω.

Proposition 2.3.12. *Let B a finite self-sufficient subalgebra of $\mathbb{M}$.*
Consider the following set of types in $S(B)$

$$\mathfrak{X} = \{\mathrm{tp}(\bar{a}/B) \mid B \leqslant \mathbb{M}, \bar{a} \subseteq \mathbb{M}_1, d(\bar{a}/B) = 0\}, \tag{2.17}$$

then $\mathrm{MR}(p)$ and $U(p)$ are finite and coincide for all type p in $\mathfrak{X}$.
The finite rank of $\mathrm{tp}(\bar{a}/B)$ coincide with the number of prealgebraic steps in a minimal decomposition of $ssc(B, \bar{a})$ over B.

Proof. As the cl_2-dimension over B of a tuple $\bar{a}$ is an invariant of the type of $\bar{a}$ over B, the family $\mathfrak{X}$ is well defined. For a fixed B and length n, $S_n(B) \cap \mathfrak{X}$ is a closed set.

By Lemma 2.3.5 each type of $\mathfrak{X}$ is isolated. Moreover if $q \in S(C)$ extends a type p of $\mathfrak{X}$ over B, for some finite set C above B, then by *ssc* we find a finite strong subspace $D_1 \leqslant \mathbb{M}_1$ such that $B \subseteq D$ and $D \subseteq \mathrm{acl}(C)$. Of course $d(\bar{a}/D) = 0$ for any realisation $\bar{a} \subseteq \mathbb{M}_1$ of q.

Therefore, up to algebraicity, the assumptions of Lemma 1.3.4 with respect to $\mathfrak{X}$ are fulfilled. Morley rank and U-rank do coincide on $\mathfrak{X}$.

We are now to prove that types in $\mathfrak{X}$ have finite rank, to do this assume $d(\bar{a}/B) = 0$ for some tuple $\bar{a}$ of $\mathbb{M}_1$ and set A equal to $ssc(B, \bar{a})$.

Assume first that A is a *minimal extension* of B. Then since $d(A/B) = d(\bar{a}/B) = 0$, then A is either algebraic or pre-algebraic over B.

In the former case, then clearly $\mathrm{MR}(\bar{a}/B) = 0$. Next we show that the type of a pre-algebraic extension A of a self-sufficient algebra B in $\mathbb{M}$, is *minimal* in the sense that it admits a unique non-algebraic extension to every set C containing B, this is equivalent for such types to have Lascar rank 1. In our case, since Morley and Lascar rank coincide, these types are actually *strongly* minimal.

That A isn't algebraic over B is Lemma 2.3.8. We may then take without loss, a subspace C_1 of $\mathbb{M}_1$ containing B_1. Since A is minimal over B and the intersection $A_1 \cap ssc(C_1)$ is strong in A_1 after Lemma 2.1.12, then either A *is contained* in $ssc(C)$ – hence algebraic over C – or $A_1 \cap ssc(C_1) = B_1$. In the latter case we have $0 \leq \delta(A/\, ssc(C)) \leq \delta(A/B) = 0$, which implies that A and $ssc(C)$ are in free composition over B (Lemma 2.2.4) and that $A + ssc(C)$ is self-sufficient in $\mathbb{M}$ (Lemma 2.1.16).

Since by Proposition 2.3.3 the isomorphism type of

$$ssc(A + C) = \langle A_1 + ssc(C_1)\rangle^M \simeq A \circledast_B ssc(C)$$

fully determines the type of A over C, this gives but only one non-algebraic type over C extending $\mathrm{tp}(A_1/B)$. That is $\mathrm{MRd}(\bar{a}/B) = (1, 1)$.

For the case in which A is not minimal over B let

$$B = A^0 \leqslant A^1 \leqslant \cdots \leqslant A^n = A$$

be a minimal decomposition of A over B as in (2.10).

Since again $d(A^{i+1}/A^i) = 0$ for each i, each section A_1^{i+1}/A_1^i is of algebraic or pre-algebraic kind.

We may now use additivity of Lascar Rank (Fact 1.3.3) and obtain

$$\mathrm{MR}(\bar{a}/B) = U(\bar{a}/B) = U(A/B) = U(A^n/A^{n-1}) + \ldots + U(A^1/A^0)$$

and conclude $\mathrm{MR}(\bar{a}/B) \leq n$.

We have shown that, types $\mathrm{tp}(\bar{a}/B)$ of tuples $\bar{a}$ of $\mathbb{M}_1$, over a strong $B_1 \subseteq \mathbb{M}_1$, such that $d(\bar{a}/B) = 0$, do have finite Morley Rank, and this rank coincides with the number of pre-algebraic steps in a minimal decomposition of $ssc(B, \bar{a})$ over B, which is a posteriori an invariant of types in $\mathfrak{X}$.

□

Proof of Theorem 2.3.11.
(*1st Claim*) $\mathrm{MR}(\mathbb{M}) = \mathrm{MR}(\mathbb{M}_1 \times \mathbb{M}_1)$ and $\mathrm{Md}(\mathbb{M}) \leq \mathrm{Md}(\mathbb{M}_1 \times \mathbb{M}_1)$.

Considering the definable map ϑ_m of Remark 2.3.10. With Fact 1.3.2(*1.*) we obtain $\mathrm{MR}(\mathbb{M}_1 \times \mathbb{M}_1) = \mathrm{MR}(\mathbb{M})$. The statement about degrees is also trivial.

(*2nd Claim*) $\mathrm{MRd}(\mathbb{M}_1) = (\omega, 1)$.

The claim will follow by showing that there is a unique generic type in the group $(\mathbb{M}_1, +)$, such type having Morley rank ω.

By the *finite* local character (cfr. Fact 1.3.5) of non-forking in totally transcendental theories, in order to compute the rank of types in T^2, it is enough to consider finite sets of parameters.

We can therefore restrict our analysis to the clopen sets

$$S_{P_1}(B) := \{p \in S_1(B) \mid P_1(x) \in p\}$$

for finite sets of parameters B in $\mathbb{M}$.

Moreover by Remark 2.3.10 and algebraicity of *ssc*, the sets B above may always be assumed to be finite strong $\mathfrak{L}_p^2$-subalgebras of $\mathbb{M}_1$.

By Remark 2.3.4, all the elements of $\mathbb{M}_1$ which are $\mathcal{cl}_2$-independent of B have all the same type over B, which we denote by p_B.

Denote by $\mathfrak{X}_B$ the set of all types $\mathrm{tp}(m/B)$ of elements m of $\mathbb{M}_1$ with $d(m/B) = 0$.

We have then

$$S_{P_1}(B) = \mathfrak{X}_B \cup \{p_B\} \tag{2.18}$$

and Morley rank of types in $\mathfrak{X}_B$ is finite by Proposition 2.3.12.

On the other hand, by Remark 2.2.12 the rich model $\mathbb{M}$ can embed arbitrarily long chains of prealgebraic extensions. This implies by Proposition 2.3.12, Morley rank of types in $\mathfrak{X}_B$ is not bounded.

As a result, we have $\mathrm{MR}(p_B) \geq \omega$ and, by Remark 1.3.1 for any formula $\psi(x)$ over B, either $\mathrm{MR}(\psi(x)) = 0$ or $\mathrm{MR}(\neg\psi(x)) = 0$. Hence $\mathrm{MR}(p_B) = \omega = \mathrm{MR}(\mathbb{M}_1)$ for any strong finite B.

When $B = \mathbf{0}$, the unique generic type $p_{\mathbf{0}}$ in $\mathbb{M}_1$ over $\varnothing$ is the type of any non-trivial element, it follows $\mathbb{M}_1$ is connected and the claim is proved.

In particular since Lascar rank is connected, then $U(p_B)$ must also be equal to ω. That is for complete types in P_1 Lascar rank and Morley rank do coincide.

(*3rd Claim*) $\mathrm{MRd}(\mathbb{M}_1 \times \mathbb{M}_1) = (\omega \cdot 2, 1)$.

It suffices once again to discuss rank of types of couples of elements in $\mathbb{M}$ over finite strong subspaces. Once again, we use $\mathscr{cl}_2$ dimension, to discern kind of types.

Let B be a fixed finite self-sufficient algebra in $\mathbb{M}$, for arbitrary elements a, b of $\mathbb{M}_1$, we have $d(a, b/B) = d(a/B, b) + d(b/B) \leq 2$.

We may therefore assume without loss of generality, one of the following three cases holds:

(1) $d(a, b/B) = 2$

(2) $d(a/B, b) = 0$ and $d(b/B) = 1$

(3) $d(a, b/B) = 0$

In the first case, a and b are in particular linearly independent over B_1 and by Lemma 2.1.16 we have $B_1 \leqslant \langle B_1, a\rangle \leqslant \langle B_1, a, b\rangle \leqslant \mathbb{M}_1$. Remark 2.3.4 implies that such pairs have all the same type over B. This type will be denoted q_B.

On the other hand, by Proposition 2.3.12, all types in (3) with $d(a, b/B) = 0$ have finite – unbounded – Morley (=Lascar) rank.

Now we have to deal with case (2) Types. We show for such types Morley rank is bounded by $\omega \cdot 2$.

Let $\bar{c}$ a tuple in $\mathbb{M}_1$ such that $ssc(B, a, b) = \langle B_1, a, b, \bar{c}\rangle^{\mathbb{M}}$, we have

$$\mathrm{MR}(a, b/B) = \mathrm{MR}(a, b, \bar{c}/B) \leq \mathrm{MR}(\varphi(x, \bar{y}, z))$$

where $\varphi(x, \bar{y}, b)$ describes the quantifier-free $\mathscr{L}^2_{Bb}$-type of $A := ssc(B, a, b)$ like in Corollary 2.3.5. The variables $x, \bar{y}$ take the places of $a, \bar{c}$.

Moreover since by (2) $d(a, \bar{c}/B, b) = 0$, let $\mathrm{MR}(a, \bar{c}/B, b) = r < \omega$. Note also that $\langle B_1, b\rangle$ is strong, and that r coincides with the number of pre-algebraic extensions in a minimal decomposition of A over $\langle B_1, b\rangle^{\mathbb{M}}$.

We want to apply Fact 1.3.2.(*2.*) to the definable map $\pi\colon \mathscr{D} \to \mathscr{E}$ where $\mathscr{D}$ denotes $\varphi(\mathbb{M}_1)$ and $\mathscr{E}$ stands for $((\exists x \exists \bar{y})\varphi)(\mathbb{M}_1)$ and π is just the projection $(x, \bar{y}, z) \mapsto z$.

By the second claim above $\mathrm{MR}(\mathscr{E})$ is at most ω and if e is an element of $\mathscr{E}$, we will prove $\mathrm{MR}(\pi^{-1}(e)) = \mathrm{MR}(\varphi(x, \bar{y}, e))$ is smaller than r.

By Remark 1.3.1 it suffices to prove $\mathrm{MR}(p(x,\bar{y})) \leq r$ for types p in $S_{x,\bar{y}}(C)$ whenever p implies $\varphi(x,\bar{y},e)$ and C is a finite self-sufficient subalgebra of $\mathbb{M}$ which contains B and e.

Let $u, \bar{v}$ realise $p(x,\bar{y})$ over C and let U denote $\langle B_1, e, u, \bar{v}\rangle^{\mathbb{M}}$.

Now $\varphi(u,\bar{v},e)$ witness $\langle B_1, e\rangle^{\mathbb{M}} \leqslant U$ and a minimal decomposition

$$\langle B_1, e\rangle^{\mathbb{M}} \leqslant U^1 \leqslant \cdots \leqslant U^n = U$$

with at most r pre-algebraic steps and $\delta(U^{i+1}/U^i) = 0$ for all i.

Since Lemma 2.1.12 implies $C_1 \cap U_1 \leqslant U_1$, then C_1 meets U_1 necessarily in some U_1^k for $1 \leq k \leq n$. Moreover $\delta(U/C) \leq \delta(U/U^k) = 0$ and then $C + U$ is self-sufficient. This yields

$$ssc(C, u, \bar{v}) = C + U = U \circledast_{U^k} C$$

and $d(u, \bar{v}/C) = 0$. Now by Lemma 2.2.14

$$C \leqslant C + U^{k+1} \leqslant \cdots \leqslant C + U^n = C + U \tag{2.19}$$

is a minimal decomposition of $C + U$ over C with

$$C + U^{i+1} \simeq U^{i+1} \circledast_{U^i} (C + U^i)$$

for all $i \geq k$ and where the minimal extension $C + U^{i+1}$ of $C + U^i$ is exactly of the same type as U^{i+1} over U^i. This means that the minimal decomposition (2.19) contains at most r pre-algebraic steps.

Since as observed before $\mathrm{MR}(u, \bar{v}/C) = \mathrm{MR}(ssc(C, u, \bar{v})/C)$ coincides with the pre-algebraic steps of a minimal decomposition between C and $ssc(C, u, \bar{v})$, this rank is bounded by r. This yields $\mathrm{MR}(\varphi(x,\bar{y},e)) \leq r$ and Fact 1.3.2 now implies $\mathrm{MR}(\phi(x,\bar{y},z)) \leq n \cdot (\omega + 1) = n \cdot \omega + n = \omega + n$.

As already pointed out in the previous claim, on the other hand $\mathbb{M}$ is rich enough to embed diagrams with rank exactly $\omega + n$ with unboundedly large $n < \omega$. As a result, by the same arguments we used above, the type q_B is the unique generic of Morley rank $\omega \cdot 2$. It follows $\mathrm{MR}(\mathbb{M}_1 \times \mathbb{M}_1) = \omega \cdot 2$.

Putting the three claims together we obtain $\mathrm{MRd}(\mathbb{M}) = (\omega \cdot 2, 1)$ and the theorem is proven.

□

Remark 2.3.13. Let $\mathbb{G} = G(\mathbb{M})$ be the $\mathfrak{N}_p^2$-group interpretable in $\mathbb{M}$ with Corollary 1.4.21. Then $\mathbb{G}$ is a connected ω-stable group of Morley rank $\omega \cdot 2$ with $Z(\mathbb{G}) = \mathbb{G}'$ and $\mathrm{MR}(\mathbb{G}_{ab}) = \mathrm{MR}(G') = \omega$.

2.3.2 Characterisation of Forking Independence

We conclude the chapter with a complete picture of forking in T^2 in terms of $c\ell_d$-independence and free amalgamation.

Recall that the self-sufficient closure $ssc^{\mathbb{M}}$ is defined on *sets* S of $\mathbb{M}$, by composing $ssc(\langle S\rangle_{\mathbb{Z}_p})$.

We now introduce a ternary relation among sets A, B and tuples $\bar{a}$ of $\mathbb{M}_1$ $\bar{a} \downfree_B A$ which will turn out to be a *irreflexive independence relation* in the sense of [Adl07].

Definition 2.3.14. For any tuple $\bar{a}$ of $\mathbb{M}_1$ and any small *sets* A and B of $\mathbb{M}_1$ *define*

$$\bar{a} \downfree_B A \quad \text{if} \quad \begin{cases} d(\bar{a}/B) = d(\bar{a}/AB) \\ ssc(B, \bar{a}) \cap ssc(AB) \subseteq \operatorname{acl}(B). \end{cases}$$

Each time $\bar{a} \downfree_B A$ holds, we say that $\bar{a}$ is $\downfree$*-independent* of A *over* B.

We extend this relation to $\mathfrak{L}^2_p$*-subalgebras* A, B of $\mathbb{M}$ by writing $\bar{a} \downfree_B A$ whenever $\bar{a} \downfree_{B_1} A_1$.

Notice that $\downfree$ satisfies Invariance as defined in Section 1.3, that is for all A, B and all $\sigma \in \operatorname{Aut}(\mathbb{M})$, $\bar{a} \downfree_B A$ iff $\bar{a}^\sigma \downfree_{B^\sigma} A^\sigma$.

We forget for the moment that our theory is totally-transcendental and prove indeed that the properties in Fact 1.3.5 are satisfied by $\downfree$-independence and, as a result, that $\downfree$ is non-forking independence among tuples and sets of $\mathbb{M}_1$. On the other hand, Remark 2.3.10 imply that forking is witnessed entirely by tuples and sets of $\mathbb{M}_1$.

Note for the moment, that $\mathscr{cl}_d$*-independece* alone cannot coincide with non-forking, for assume B is a strong algebra in $\mathbb{M}$ and $\bar{a}$ is a tuple of $\mathbb{M}_1$, assume that $ssc(B, \bar{a})$ decomposes into $B \leqslant A \leqslant ssc(B, \bar{a})$ where $ssc(B, \bar{a})$ is minimal algebraic over A and A is a pre-algebraic extension of B, then $d(\bar{a}/A) = d(\bar{a}/B) = 0$ but, as proven in Theorem 2.3.11, $\mathrm{MR}(\bar{a}/A) = 0 < \mathrm{MR}(\bar{a}/B) = 1$.

This is essentially the only obstruction, since in the collapsed structure prealgebraic extensions are forced to be algebraic: and the geometric closure and the algebraic coincide.

The following proposition shows that $\downfree$-independence is expressible by means of free composition of algebras.

Proposition 2.3.15. *For any tuple $\bar{a}$ of $\mathbb{M}_1$ and subalgebras A, B of $\mathbb{M}$ if C denotes $\langle ssc(B_1, \bar{a}) \cap ssc(A_1 + B_1)\rangle^{\mathbb{M}}$, then the following holds:*

(i.) *$d(\bar{a}/B) = d(\bar{a}/AB)$ implies $ssc(A + B, \bar{a}) \simeq ssc(B, \bar{a}) \circledast_C ssc(A + B)$*

(ii.) *assume both that $ssc(A + B, \bar{a}) \simeq ssc(B, \bar{a}) \circledast_C ssc(A + B)$ and $C_1 \subseteq \operatorname{acl}(B_1)$ hold, then $\bar{a} \downfree_B A$.*

Proof. (i.) For sake of simplicity we will assume, that $B \leqslant A \leqslant \mathbb{M}$. Let also C_1 denote $ssc(B_1, \bar{a}) \cap A_1$. Then we have to show $ssc(A, \bar{a}) \simeq ssc(B, \bar{a}) \circledast_C A$ whenever $d(\bar{a}/A) = d(\bar{a}/B)$.

We first prove that $\delta_2(ssc(B,\bar{a})/C) = \delta_2(ssc(B,\bar{a})/A)$. We have in fact, by Lemma 2.1.4 and 2.1.17.(i),(iii)

$$\begin{aligned} d(\bar{a}/A_1) \leq d(ssc(B_1,\bar{a})/A_1) \leq \\ \leq \delta_2(ssc(B_1,\bar{a})/A_1) \leq \delta_2(ssc(B_1,\bar{a})/C_1) = \\ = \delta_2(ssc(C_1,\bar{a})/C_1) = d(\bar{a}/C_1) \leq d(\bar{a}/B_1). \quad (2.20) \end{aligned}$$

This implies on one side, by the hypothesis and Lemma 2.2.4 that $ssc(B,\bar{a})$ is in free composition with A (over C). This means just that $ssc(B,\bar{a})+A = \langle ssc(B_1,\bar{a}) + A_1\rangle^{\mathbb{M}} \simeq ssc(B,\bar{a}) \circledast_B A$.

On the other hand, since by (2.20), $d(ssc(B_1,\bar{a})/A_1) = \delta_2(ssc(B_1,\bar{a})/A_1)$, Lemma 2.1.17.(iv) again, implies that $ssc(B_1,\bar{a}) + A_1$ is self-sufficient in $\mathbb{M}_1$, and therefore $ssc(B_1,\bar{a}) + A_1 = ssc(A_1,\bar{a})$.

(ii.) Assume now $ssc(A,\bar{a}) \simeq ssc(B,\bar{a}) \circledast_C A$.

Then in particular $ssc(B_1,\bar{a}) + A_1 = ssc(A_1,\bar{a}) \leqslant \mathbb{M}_1$. The second hypothesis in (ii.) implies, with Remark 2.3.9, $\delta_2(ssc(B_1,\bar{a})/B_1) = \delta_2(ssc(B_1,\bar{a})/C_1)$, hence by Lemmas 2.2.4 and 2.1.17.(iv) we have

$$\begin{aligned} d(\bar{a}/B_1) = \delta_2(ssc_2(B_1,\bar{a})/B_1) = \\ = \delta_2(ssc_2(B_1,\bar{a})/C_1) = \delta_2(ssc_2(B_1,\bar{a})/A_1) = \\ = d(ssc_2(B_1,\bar{a})/A_1) = d(\bar{a}/A). \end{aligned}$$

□

Remark 2.3.16. It is clear that we may extend the $\downfree$-relation to sets or subalgebras in the first entry by defining $A \downfree_B C$ to hold, whenever all tuples $\bar{a}$ from A are $\downfree$-independent of C over B.

By

The correspondence between the relation $\downfree$ and free amalgams allows us to prove that $\downfree$-independence satisfies a *finite* instance of Boundedness property, described in section 1.3. This is shown in the next proposition, along with *finite* Local Character for $\downfree$ (cfr. Fact 1.3.5).

We call a type $p \in S(A)$ with $A \supseteq B$, a $\downfree$-*independendent* extension of $p{\restriction}_B$ if for any $\bar{a}$ realising p over A, $\bar{a} \downfree_B A$ holds.

Lemma 2.3.17. *For any $\mathfrak{L}_p^2$-subalgebra B of $\mathbb{M}$ and $\bar{a}$ in $\mathbb{M}_1$.*

(i.) *There is a finite subset $B^0 \subseteq B$ such that $\bar{a} \downfree_{B^0} B$.*

(ii.) *For any $A \supseteq B$, there are at most finitely many distinct orbits under $\mathrm{Aut}_A(\mathbb{M})$ in the set of all $\bar{a}'$ with $\bar{a}' \downfree_B A$ and $\bar{a}' \equiv_B \bar{a}$. That is, at most finitely many $\downfree$-independent extensions of $\mathrm{tp}(\bar{a}/B)$ to A.*

Proof. (i.) That $\downarrow$ satisfies (i.) is precisely statement (v) of Lemma 2.1.17.

(ii.) Assume $\bar{a}' \downarrow_B A$, for some $\bar{a}'$ in $\mathbb{M}$.

This gives by Proposition 2.3.15 that $ssc(A, \bar{a}') \simeq ssc(B, \bar{a}') \circledast_C ssc(A)$, where $C_1 = ssc(B_1, \bar{a}') \cap ssc(A_1) \subseteq \mathrm{acl}(B_1)$.

On the other hand, since $\bar{a}' \equiv_B \bar{a}$, Proposition 2.3.3 implies that $ssc(B, \bar{a}') \simeq ssc(B, \bar{a})$ via an isomorphism which maps $\bar{a}$ onto $\bar{a}'$ and fixes B.

This implies that the quantifier-free type of $ssc(A, \bar{a}')$, depends only on the choices for the subspace C_1 between B_1 and $ssc(B_1, \bar{a}') \cap \mathrm{acl}(B_1)$, and these are just in a finite number.

Proposition 2.3.3 again, give only finitely many representatives $\bar{a}'$ in $\mathbb{M}_1$ modulo A-conjugacy, which are $\downarrow$-independent of A over B.

□

As expected, a type over a algebraically closed set B is $\downarrow$-*stationary* in the sense of the following

Corollary 2.3.18. *Assume a tuple $\bar{a} \subseteq \mathbb{M}_1$ and a subalgebra B are given. A type $p = \mathrm{tp}(\bar{a}/B)$ is* $\downarrow$-stationary (that is, it admits a unique $\downarrow$-independent extension q to any algebra $A \supseteq B$ of $\mathbb{M}$) *whenever $ssc(B_1, \bar{a}) \cap \mathrm{acl}(B_1) = B_1$.*

We prove the remaining non-forking properties in the following proposition.

Proposition 2.3.19. *The relation $\downarrow$ introduced by Definition 2.3.14 satisfies the following properties*

(Transitivity) *for all $C \subseteq B \subseteq A$, from $\bar{a} \downarrow_C B$ and $\bar{a} \downarrow_B A$ follows $\bar{a} \downarrow_C A$,*

(Monotony) *if $\bar{a} \downarrow_C A$ and $C \subseteq B \subseteq A$ then $\bar{a} \downarrow_C B$.*

(Existence) *for any $\bar{a}$ and $B \supseteq C$ there exists a tuple $\bar{a}'$ in $\mathbb{M}_1$ with $\bar{a}' \equiv_C \bar{a}$ such that $\bar{a}' \downarrow_C B$.*

Proof. To prove Transitivity, let $A \supseteq B \supseteq C$ be (strong) subalgebras of $\mathbb{M}$, and $\bar{a}$ is a tuple of $\mathbb{M}_1$ with both $\bar{a} \downarrow_B A$ and $\bar{a} \downarrow_C B$. This gives $d(\bar{a}/A) = d(\bar{a}/B) = d(\bar{a}/C)$ and hence we have to show $ssc(C_1, \bar{a}) \cap A_1 \subseteq \mathrm{acl}(C_1)$.

Since $\bar{a}$ is $\downarrow$-independent of A over B

$$ssc(C_1, \bar{a}) \cap A_1 = ssc(C_1, \bar{a}) \cap ssc(B_1, \bar{a}) \cap A_1 \subseteq ssc(C_1, \bar{a}) \cap \mathrm{acl}(B_1).$$

If D_1 denotes $ssc(C_1, \bar{a}) \cap B_1$, then $ssc(C_1, \bar{a}) = ssc(D_1, \bar{a})$ and $\bar{a} \downarrow_C B$ implies $D_1 \subseteq \mathrm{acl}(C_1)$ and with Proposition 2.3.15, that $ssc(D, \bar{a})$ is in free composition with B over D, moreover $ssc(B, \bar{a}) \simeq ssc(D, \bar{a}) \circledast_D B$.

Lemma 2.2.14 and Remark 2.3.9 now imply $\mathrm{acl}(B_1) \cap ssc(B_1, \bar{a})$ is forced to meet any minimal decomposition of $ssc(D, \bar{a})$ over D necessarily within the first adjacent minimal algebraic extensions of D.

This means $ssc(D_1, \bar{a}) \cap \mathrm{acl}(B_1) \subseteq \mathrm{acl}(D_1)$ and since $D_1 \subseteq \mathrm{acl}(C_1)$, we may conclude $ssc(C_1, \bar{a}) \cap A_1 \subseteq \mathrm{acl}(C_1)$ as desired.

While Monotony is trivial to prove, to show Existence let $\bar{a}$ be a tuple and $B \supseteq C$ algebras, which might – without loss of generality be assumed strong in $\mathbb{M}$. Denote by A the self-sufficient closure $ssc(C, \bar{a})$.

By collecting all divisors[8] of C in A_1 which are realised (cfr. Definition 2.2.11) in B, we can find $\widetilde{C}$, such that $C \subseteq \widetilde{C} \subseteq \mathrm{acl}(C_1)$, $A \geqslant \widetilde{C} \leqslant B$ and there is no divisor of $\widetilde{C}$ in A_1 which is realised in B.

Take an isomorphic copy $\widetilde{A}$ of A and denote by $\tilde{a}$ the image of $\bar{a}$ inside $\widetilde{A}$. Now denote by $\widetilde{B}$, the free amalgam $B \circledast_{\widetilde{C}} \widetilde{A}$ of B and $\widetilde{A}$ over $\widetilde{C}$. By Lemma 2.2.18 follows, that $\widetilde{B}$ inherits $\Sigma^2(2)$ and hence $\widetilde{B}$ is a finite strong extension of B which is in $\widetilde{\mathcal{K}}_2$. By richness of $\mathbb{M}$ after Remark 2.3.7 we can find an embedding σ of $\widetilde{B}$ into $\mathbb{M}$ over B such that $\widetilde{B}^\sigma \leqslant \mathbb{M}$.

Now since $\widetilde{A} \leqslant \widetilde{B}$, $\widetilde{A}$ coincides with $ssc^{\widetilde{B}}(C, \tilde{a})$ and hence if $\bar{a}'$ denotes the image in $\mathbb{M}_1$ of $\tilde{a}$ under σ, we have $\widetilde{A}^\sigma = ssc^{\mathbb{M}}(C, \bar{a}')$.

On the other hand $\widetilde{B}$ must coincide with the the self-sufficient closure (in $\widetilde{B}$) of B and $\tilde{a}$. This gives analogously $\widetilde{B}^\sigma = ssc^{\mathbb{M}}(B, \bar{a}') \simeq B \circledast_{\widetilde{C}} ssc^{\mathbb{M}}(C, \bar{a}')$. With (ii.) of Proposition 2.3.15 we obtain $\bar{a}' \mathop{\smile\!\!\!\!\!\!|}_C B$. Then of course $\bar{a}' \equiv_C \bar{a}$, since $ssc^{\mathbb{M}}(C, \bar{a}') \simeq \widetilde{A} \simeq ssc^{\mathbb{M}}(C, \bar{a})$.

□

Putting Lemma 2.3.17 and Proposition 2.3.19 together with Fact 1.3.5 we reobtain (ω-) stability of T_2 and in particular:

Corollary 2.3.20. *On the sets of $\mathbb{M}_1$ forking independence and $\mathop{\smile\!\!\!\!\!\!|}$-independence coincide. This is*

$$\bar{a} \mathop{\smile\!\!\!\!\!\!|}_B A \iff \bar{a} \mathop{\smile\!\!\!\!\!\!|}^f_B A$$

for all $\bar{a}, A, B \subseteq \mathbb{M}_1$.

Remark. For any given *set* B in $\mathbb{M}_1$, the forking geometry on the generic type $p = p_{ssc(B)}$ in $\mathbb{M}_1$ over $ssc(B)$ defined in Theorem 2.3.11, is exactly the $c\ell_d$-geometry. That is to say, for any $a, \bar{b}$ in $p(\mathbb{M}_1)$ we have

$$a \in c\ell_d(\bar{b}/B) \iff a \not\mathop{\smile\!\!\!\!\!\!|}_B \bar{b}$$

Proof. By Remark 2.1.18 and the definition of $\mathop{\smile\!\!\!\!\!\!|}$, we may assume B is a self-sufficient subspace of $\mathbb{M}_1$.

For the left-to-right implication, a routine induction argument with monotonicity of forking let us assume $\bar{b}$ is a singleton $b \in p_B(\mathbb{M}_1)$. Now $a \in c\ell_d(B, b) \setminus c\ell_d(B)$ gives – by exchange – $b \in c\ell_d(B, a)$, hence $d(a/B) = 1 > 0 = d(a/B, \bar{b})$.

Now if $a \not\mathop{\smile\!\!\!\!\!\!|}_B \bar{b}$, then either $d(a/B, \bar{b}) = 0$ or $ssc(B, a) = \langle B, a \rangle_{\mathbb{Z}_p} \subseteq ssc(B, \bar{b}) \subseteq c\ell_d(B, \bar{b})$.

□

[8] or simply by taking the relative algebraic closure $\mathrm{acl}(C_1) \cap A_1$

2.3.3 Around weak elimination of Imaginaries and *CM*-Triviality

We conclude this chapter with some results about identifying canonical bases for types over models of T^2.

In this section, a distinguished non-trivial element of $\mathbb{M}_1$ will be added to the signature as parameter. With Remark 2.3.10, this implies we can actually work with tuples from $\mathbb{M}_1$ only.

The result which follows provides a notion of *weak* canonical base for types of self-sufficient tuples over models and inspired a program about showing CM-triviality for T^2. This is now a work in progress which is not accomplished in this thesis.

For any strong $\mathfrak{L}_p^2$-subalgebra H of $\mathbb{M}$, we call a tuple $\bar{a} \subseteq \mathbb{M}_1$ a *strong tuple over* H, if $\bar{a}$ is linearly independent of H_1 and if $\langle H_1, \bar{a}\rangle_{\mathbb{Z}_p} \leqslant \mathbb{M}_1$. For the definition of canonical bases we refer to Section 1.3.

Lemma 2.3.21. *We fix a constant c to the language $\mathscr{L}^2$, interpreted by an element of $\mathbb{M}_1$ different from $\mathbf{0}$.*

For any model M of T_c^2, and any strong tuple $\bar{a}$ over M, there is a finite strong $\mathfrak{L}_p^2$-subalgebra C of M, such that

$$C \subseteq \operatorname{acl}^{eq}(Cb(\bar{a}/M)) \text{ and } Cb(\bar{a}/M) \subseteq \operatorname{dcl}^{eq}(C).$$

Proof. Let p denote $\operatorname{tp}(\bar{a}/M)$ and $\bar{a}$ be $a_0, \ldots, a_{n-1}$.

Let $\operatorname{Aut}_{\{M\}}(\mathbb{M})$ denote the group of all automorphism of the monster $\mathbb{M}$, which leave M invariant.

If σ is an automorphism of $\mathbb{M}$, whose restriction to M fixes the type p, then $\bar{a}^\sigma \equiv_M \bar{a}$. Since M is small, the strong homogeneity of the monster implies, that the action *on* M of the stabiliser of the type p in $\operatorname{Aut}(M)$ coincides with the action on M under the pointwise stabiliser of $\bar{a}$ in $\operatorname{Aut}_{\{M\}}(\mathbb{M})$. We will first find a finite subspace of M which is invariant under all $\sigma \in \operatorname{Aut}_{\{M\}}(\mathbb{M})$ with $\bar{a}^\sigma = \bar{a}$.

Let $(\rho_i)_{i<m}$ be a set in $\bigwedge^2\langle M_1, \bar{a}\rangle_{\mathbb{Z}_p}$ linearly independent over $\bigwedge^2 M_1$ which is a basis of $R^2_{\mathbb{M}}(M, \bar{a})$ over $R^2(M)$. Since M is self-sufficient, then $m \leq n$.

For all $i < m$, we find a tuple $\bar{b}_i = ({b_i}^0, \ldots, {b_i}^{n-1})$ of length n, of not necessarily linearly independent elements of M_1 such that

$$\rho_i = \alpha_i + \beta_i + \gamma_i \tag{2.21}$$

where each $\alpha_i = \alpha_i(\bar{a})$ is in $\bigwedge^2\langle \bar{a}\rangle_{\mathbb{Z}_p}$ and every γ_i – when nontrivial – lays in $\bigwedge^2 M_1 \backslash R^2(M)$ for all $i < m$. Moreover, all nontrivial β_i are given by

$$\beta_i = \beta(\bar{a}, \bar{b}_i) = \sum_{k<n} [a_k, {b_i}^k] \in \bigwedge^2\langle M_1, \bar{a}\rangle_{\mathbb{Z}_p} \setminus \bigwedge^2\langle \bar{a}\rangle_{\mathbb{Z}_p} + \bigwedge^2 M_1.$$

Let now σ be an automorphism in $\operatorname{Aut}_{\{M\}}(\mathbb{M})$ which fixes $\bar{a}$ pointwise. Let $\widehat{\sigma}$ be the graded Lie isomorphism induced by $\sigma\restriction_{\mathbb{M}_1}$ on the free graded algebra $L^2(\mathbb{M}_1) = \mathbb{M}_1 \oplus \bigwedge^2 \mathbb{M}_1$ like in Lemma 1.4.13.

With this notation, as $\sigma(\langle M, \bar{a}\rangle^{\mathbb{M}}) = \langle M, \bar{a}\rangle^{\mathbb{M}}$, it follows $\widehat{\sigma}\left(R^2(M_1, \bar{a})\right) = R^2(M_1, \bar{a})$. Moreover since $(\rho_i)_{i<m}$ is a basis of $R^2(\bar{a}/M)$, for all i we have

$$\rho_i{}^{\widehat{\sigma}} - \sum_{j<m} s_j \rho_j = \mu \tag{2.22}$$

for some μ in $\bigwedge^2 M_1$ and s_j in $\mathbb{Z}_p$. On the other side

$$\rho_i{}^{\widehat{\sigma}} = \alpha_i(\bar{a}) + \beta(\bar{a}, \bar{b}_i)^{\widehat{\sigma}} + \gamma_i^{\widehat{\sigma}} = \alpha_i(\bar{a}) + \beta(\bar{a}, \bar{b}_i{}^{\sigma}) + \gamma_i^{\widehat{\sigma}}$$

where

$$\beta(\bar{a}, \bar{b}_i^{\sigma}) = \sum_{k<n} [a_k, \sigma(b_i{}^k)] \tag{2.23}$$

Now (2.22) becomes

$$\alpha_i - \sum_{j<m} s_j \alpha_j + \beta_i^{\widehat{\sigma}} - \sum_{j<m} s_j \beta_j = \mu - \gamma_i^{\widehat{\sigma}} + \sum_{j<m} s_j \gamma_j \ \in \bigwedge^2 M_1 \tag{2.24}$$

and since $\langle \alpha_i, \beta_i, \beta_i^{\widehat{\sigma}} \mid i < m\rangle_{\mathbb{Z}_p} \cap \bigwedge^2 M_1 = \mathbf{0}$, one has

$$\alpha_i + \beta(\bar{a}, \bar{b}_i)^{\widehat{\sigma}} = \sum_{j<m} s_j(\alpha_j + \beta(\bar{a}, \bar{b}_j)).$$

Now by the same arguments, we get

$$\beta(\bar{a}, \bar{b}_i^{\sigma}) = \sum_{j<m} s_j \beta(\bar{a}, \bar{b}_j).$$

Hence in particular

$$\sum_{k<n} [a_k, \sigma(b_i{}^k)] = \sum_{j<m} s_j \beta(\bar{a}, \bar{b}_j) = \sum_{k<n} [a_k, \sum_{j<m} s_j b_j{}^k]. \tag{2.25}$$

Now by Fact 1.4.10, in $\bigwedge^2 \mathbb{M}_1$ we have $[a_k, M_1] \cap \sum_{j \neq k} [a_j, M_1] = \mathbf{0}$. Therefore (2.23) and (2.25) imply for all k, $[a_k, \sigma(b_i{}^k)] = \sum_{k<n} [a_k, \sum_{j<m} s_j b_j{}^k]$ in $\bigwedge^2 \mathbb{M}_1$.

This yields $\sigma(b_i{}^k) = \sum_{j<m} s_j b_j{}^k$ and hence $\sigma(\langle \bar{b}_i \mid i < m\rangle_{\mathbb{Z}_p}) \subseteq \langle \bar{b}_i \mid i < m\rangle_{\mathbb{Z}_p}$.

If we denote by $\bar{b}$ the tuple $\bar{b}_0, \ldots, \bar{b}_m$, then $\bar{b} \subseteq M_1$ and $(\langle \bar{b}\rangle_{\mathbb{Z}_p})^{\sigma} \subseteq \langle \bar{b}\rangle_{\mathbb{Z}_p}$.

Let now Γ_i denote the image of γ_i in M_2 modulo $R^2(M)$, that is $\Gamma_i = \gamma_i + R^2(M)$ for all i. Since, by (2.22) and (2.24) we have

$$R^2(M) \ni \rho_i{}^{\widehat{\sigma}} - \sum_{j<m} s_j \rho_j = \gamma_i^{\widehat{\sigma}} - \sum_{j<m} s_j \gamma_j,$$

we deduce

$$\Gamma_i{}^{\sigma} = \gamma_i^{\widehat{\sigma}} + R^2(M) = \sum s_j \gamma_j + R^2(M) = \sum s_j \Gamma_j$$

and hence, if $\overline{\Gamma}$ denotes the tuple $(\Gamma_0, \ldots, \Gamma_m)$, we get $\langle\overline{\Gamma}\rangle^\sigma \subseteq \langle\overline{\Gamma}\rangle$.

Till now we have shown, that for any automorphism σ of M, if σ fixes the type p, then σ leaves both spaces $\langle\bar{b}\rangle_{\mathbb{Z}_p}$ and $\langle\overline{\Gamma}\rangle_{\mathbb{Z}_p}$ invariant.

As $Cb(\bar{a}/M)$ doesn't change by passing to non-forking extensions of p, we may assume M is ω-saturated. As $Cb(\bar{a}/M)$ is a finite imaginary, and $\langle\bar{b}\rangle_{\mathbb{Z}_p}$ and $\langle\overline{\Gamma}\rangle_{\mathbb{Z}_p}$ are finite subspaces, they lay in $\mathrm{acl}^{eq}(Cb(p))$.

Now since M is a model, by means of $\Sigma^2(4)$ we can find c_i in M_1 such that $[c, c_i] = \Gamma_i$ in M for all $i < m$. This means $[c, c_i]$ can play the role of γ_i in $\bigwedge^2 M_1$. Moreover, by Remark 2.3.10 if the tuple $\bar{c}$ collects all the c_i, we have $\bar{c} \subseteq \mathrm{acl}(\overline{\Gamma})$ – c is the constant added to the language.

Take $C = \langle C_1\rangle^M$ with $C_1 = ssc(\bar{b}, \bar{c}, c)$, then on one side $C \subseteq \mathrm{acl}(\bar{b}, \bar{c})$ and hence $C \subseteq \mathrm{acl}^{eq}(Cb(p))$.

On the other hand, by construction we have $\delta(\bar{a}/C) = \delta(\bar{a}/M)$. Therefore, as C is strong in M and $\bar{a}$ is a strong tuple over M, Lemma 2.1.17 implies

$$d(\bar{a}/M) = \delta(\bar{a}/M) = \delta(\bar{a}/C) \geq d(\bar{a}/C) \geq d(\bar{a}/M)$$

and hence $d(\bar{a}/C) = d(\bar{a}/M)$.

This also yields – with Lemma 2.1.17 again – $M + \langle C_1, \bar{a}\rangle \leqslant \mathbb{M}$ and $ssc(C_1, \bar{a}) = \langle C_1, \bar{a}\rangle_{\mathbb{Z}_p}$. Now since $\bar{a}$ is linearly independent over M_1, we have $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid\,}_C M$ and hence $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid\,}^f_C M$.

Now since M is a model, $\mathrm{acl}(C_1) \subseteq M$ and this yields with Corollary 2.3.18, that $\mathrm{tp}(\bar{a}/C)$ is stationary. Now Fact 1.3.8 (2.) implies $Cb(p) \subseteq \mathrm{dcl}^{eq}(C)$.

For such strong tuples $\bar{a}$ over M, we have shown

$$\begin{cases} C \subseteq \mathrm{acl}(Cb(\bar{a}/M)) \\ Cb(\bar{a}/M) \subseteq \mathrm{dcl}^{eq}(C). \end{cases} \tag{2.26}$$

$\square$

Compared to Lemma 1.3.9, we have obtained the condition for the weak elimination only with respect to types of strong tuples.

We denote with $C_{M,\bar{a}}$ the finite self-sufficient subalgebra of M as in (2.26) above, relative to the tuple $\bar{a}$ and the model M.

Remark that to obtain a finite subspace of M with the first property of (2.26), it is enough for the tuple $\bar{a}$ to be linearly independent over M_1. On the contrary, by δ-calculus alone (Lemma 2.1.17) a finite object with the second property of (2.26) is found for any tuple over a model – this is the finite character of forking, common to all totally transcendental structures. Starting with a strong tuple over M ensures the existence of a finite subalgebra with both properties.

This said, we can sketch a strategy for proving CM-triviality as follows. By Fact 1.3.12 one can test CM-triviality (Definition 1.3.11) by means of *real* tuples over *models*. Moreover this property for T^2_c implies the result for T^2.

So *take a tuple* $\bar{d}$ *from* $\mathbb{M}_1$ *and models* $M \preccurlyeq N$ *with* $\mathrm{acl}^{eq}(M, \bar{d}) \cap N = M$. *We have to show* $Cb(\bar{d}/M) \subseteq \mathrm{acl}^{eq}(Cb(\bar{d}/N))$.

Let $\bar{a}$ be a strong tuple over M with $ssc(M, \bar{d}) = \langle M, \bar{a}\rangle$ and $\bar{a}'$, a strong tuple over N with $ssc(N, \bar{d}) = \langle N, \bar{a}'\rangle$.

We have firstly $Cb(\bar{d}/M) \subseteq \mathrm{acl}^{eq}(C_{M,\bar{a}})$, by Fact 1.3.10 and (2.26) above.

Now by the hypothesis follows $\langle M_1, \bar{a}\rangle_{\mathbb{Z}_p} \cap N_1 = M_1$ and hence we may assume $\bar{a} \subseteq \bar{a}'$. For the same reason we also have that $R^2(\bar{a}/M)$ embeds into $R^2(\bar{a}'/N)$ and – by the proof of Lemma 2.3.21 – we may build $C_{N,\bar{a}'}$ with $C_{M,\bar{a}} \subseteq C_{N,\bar{a}'}$.

Up to now, we used indeed the definition of CM-triviality *relative to ssc( ) over the theory of* $\mathbb{Z}_p$*-vectorspaces* contained in the paper of Wagner, Blossier and Martin-Pizarro [BWMP10].

We now believe, there exists in general a $\varnothing$-definable equivalence relation ϵ, such that $C_{N,\bar{a}'}/\epsilon$ is eq-interalgebraic with $Cb(\bar{d}/N)$[9].

This should fill in the gap and get $Cb(\bar{d}/M)$ contained in $\mathrm{acl}^{eq}(Cb(\bar{d}/N))$.

[9]If $C_{N,\bar{a}'}$ is $ssc(\bar{b}, \bar{c}, c)$ as in Theorem 2.3.21, then we can prove $\bar{b} \subseteq \mathrm{acl}^{eq}(Cb(\bar{d}/N))$ and the eq-sort ϵ should control the orbits of the tuple $\bar{c}$ under automorphism which fix $\mathrm{tp}(\bar{d}/N)$.

3 Deficiencies in Higher Class

We are going to develop a new tool which permits an inductive approach to the construction of deficiencies for $\mathfrak{L}_p^c$-algebras ($p > c$) in a nilpotency class c higher than 2. The main results and details concern the case $c = 3$, for which a predimension-like approach is adopted.

3.1 A "free lift" Functor

Fix a prime p grater than c and assume $M = M_1 \oplus \cdots \oplus M_c$ is a graded Lie algebra of $\mathfrak{L}_p^c$ as defined in Definition 1.4.8 of Section 1.4, in particular $M = \langle M_1 \rangle$.

The c^{th}-homogeneous component M_c of M coincides with the ideal $\gamma_c(M)$ of M – the c^{th} term of the lower central series. If we denote by M_* the quotient M/M_c, then M_* is again generated by M_1 (modulo M_c) and $M_* \simeq M_1 \oplus \cdots \oplus M_{c-1}$. That is $M_* \in \mathfrak{L}_p^{c-1}$ and we can refer to M_* as the *truncation* of M to $\mathfrak{L}_p^{c-1}$. We denote by $*$ the resulting functor of $\mathfrak{L}_p^c$ into $\mathfrak{L}_p^{c-1}$.

For a given M in $\mathfrak{L}_p^c$, with abuse of the above notation, we denote again by $*$ the canonical epimorphism of M onto M_*. Moreover, for any m in M, we set $m_* = m* = m + M_c$.

In Section 1.4, we denoted by $L^n(X)$ the free n-nilpotent Lie $\mathbb{Z}_p$-algebra over the set X. This is in particular a free object of $\mathfrak{L}_p^n$. Moreover any A in $\mathfrak{L}_p^n$, is an epimorphic image of $L^n(A_1)$.

Now for an algebra M in $\mathfrak{L}_p^{c-1}$, let R be the homogeneous ideal of $L^{c-1}(M_1)$ which gives the presentation $\langle M_1 \mid R \rangle$ of M, as defined in section 1.4.

We identify $L^{c-1}(M_1)$ with the subspace $L_1 \oplus \cdots \oplus L_{c-1}$ of $L = L^c(M_1)$, and denote by ι the $\mathbb{Z}_p$-linear embedding of $L^{c-1}(M_1)$ into $L^c(M_1)$.

Let $\mathbb{J}(M)$ denote the ideal of $L^c(M_1)$ generated by $\iota(R)$.

Since R is an ideal of $L^{c-1}(M_1)$, in the notation of section 1.4, we have $\mathbb{J}(M) = \langle \iota(R) \rangle_{\mathsf{id}} = \langle \iota(R), [\iota(R), L^c(M_1)] \rangle_{\mathbb{Z}_p}$. Notice that $\mathbb{J}(M)$ is homogeneous with $\mathbb{J}(M)_1 = \mathbf{0}$ (cfr. (3.3) below).

Definition 3.1.1. For any M of $\mathfrak{L}_p^c$, define F_M to be the quotient $L^c(M_1)/\mathbb{J}(M)$ and call *free lift*, the map

$$\begin{aligned} \mathsf{fl} \colon \mathfrak{L}_p^{c-1} &\longrightarrow \mathfrak{L}_p^c \\ M &\longmapsto F_M = \langle M_1 \mid \mathbb{J}(M) \rangle. \end{aligned} \tag{fl}$$

Proposition 3.1.2. *For any algebra $M = \langle M_1 \mid R \rangle$ of $\mathfrak{L}_p^{c-1}$ one has $(F_M)_* \simeq M$ and we adopt the convention to identify $(F_M)_*$ with M, coherently with the choice for $(F_M)_1$ to be M_1.*

For any N in $\mathfrak{L}_p^c$, if φ is an $\mathfrak{L}_p^{c-1}$-morphism of M into N_*, then there exists a unique $\mathfrak{L}_p^c$-morphism $\widetilde{\varphi}$ of F_M into N such that $\widetilde{\varphi}* = *\varphi$.

$$\begin{tikzcd} F_M \arrow[r, "\widetilde{\varphi}"] \arrow[d, "*"] & N \arrow[d, "*"] \\ M \arrow[r, "\varphi"] & N_* \end{tikzcd} \tag{3.1}$$

Moreover, the map $\varphi \mapsto \widetilde{\varphi}$ yelds a bijection

$$\mathrm{Hom}_{\mathfrak{L}_p^{c-1}}(M, N_*) \to \mathrm{Hom}_{\mathfrak{L}_p^c}(\mathsf{fl}(M), N) \tag{3.2}$$

for any M in $\mathfrak{L}_p^{c-1}$ and N in $\mathfrak{L}_p^c$.

Proof. Let L denote $L^c(M_1)$. Then $L_c = \gamma_c(L)$ and $\mathbb{J}(M)$ are ideals of L such that $F_M = L/\mathbb{J}(M)$ and $L/L_c \simeq L^{c-1}(M_1)$. We have the following isomorphisms of Lie algebras[1]

$$(F_M)_* \simeq \frac{L}{L_c + \mathbb{J}(M)} \simeq \frac{L/L_c}{(L_c + \mathbb{J}(M))/L_c}$$

Moreover, as R is homogeneous and equals $R_2 + \cdots + R_{c-1}$, we have

$$\mathbb{J}(M) = \iota(R) \oplus \sum_{i=2}^{c-1} [R_i, L_{c-i}] \tag{3.3}$$

where the right direct summand is contained in L_c.

Hence – as algebras – $(L_c + \mathbb{J}(M))/L_c \simeq \mathbb{J}(M)/(\mathbb{J}(M) \cap L_c) \simeq R$ and therefore $(F_M)_* \simeq_{\mathfrak{L}_p^{c-1}} L^{c-1}(M_1)/R = M$. The first assertion is proved.

If we interpret M and N_* as objects of $\mathfrak{L}_p^c$, we obtain the presentations $\mu\colon L^c(M_1) \to M$ and $\eta\colon L^c(N_1) \to N_*$. For any morphism φ of M into N_*, we obtain a unique $\mathfrak{L}_p^c$-morphism $\widehat{\varphi}$ of $L^c(M_1)$ into $L^c(N_1)$ by means of Lemma 1.4.13, with the property $\widehat{\varphi}\eta = \mu\varphi$.

Through the canonical $\mathbb{Z}_p$-embedding ι, we identify $L^{c-1}(M_1)$ with a subspace of $L^c(M_1)$ as described above. If we denote by R the kernel of the restriction of μ to $L^{c-1}(M_1)$, we obtain $\ker(\mu) = R \oplus \gamma_c(L^c(M_1))$ and $L^{c-1}(M_1)$ presents M modulo R.

[1] In the row below + is the ordinary *sum* between subalgebras and ideals of a Lie algebra.

It follows $\mathbb{J}(M)$ is the ideal of $L^c(M_1)$ generated by R. In particular $\mathbb{J}(M) \subseteq \ker(\mu)$ and, if $\pi_{\mathbb{J}}$ presents F_M as a quotient of $L^c(M_1)$, then $\pi_{\mathbb{J}}* = \mu$.

On the other hand if ϵ_N presents N from $L^c(N_1)$, then $\epsilon_N* = \eta$.

If now $w \in R$, then $(w\,\widehat{\varphi}\epsilon_N)_* = w\,\widehat{\varphi}\eta = w\,\mu\varphi = \mathbf{0}$. But this means $w\,\widehat{\varphi}\epsilon_N$ lays inside $N_c \cap R^{\widehat{\varphi}\epsilon_N}$ which is trivial by a matter of weight and hence $w\,\widehat{\varphi}\epsilon_N = \mathbf{0}$.

This yields that $\mathbb{J}(M) \subseteq \ker(\widehat{\varphi}\epsilon_N)$ and hence we *define* $\widetilde{\varphi}$ as the quotient of $\widehat{\varphi}\epsilon_N$ modulo $\mathbb{J}(M)$, that is $\bar{w} \mapsto w\,\widehat{\varphi}\epsilon_N$ for $w \in L^c(M_1)$.

Any other map φ' of F_M to N with $\varphi'* = *\varphi$ fits in the diagram below in the place of $\widetilde{\varphi}$. In particular $\widehat{\varphi}\epsilon_N = \pi_{\mathbb{J}}\varphi'$ and hence $\varphi' = \widetilde{\varphi}$.

$$\begin{tikzcd}
 & L^c(M_1) \arrow[rr, "\widehat{\varphi}"] \arrow[d, "\pi_{\mathbb{J}}"] \arrow[ddl, "\mu"'] & & L^c(N_1) \arrow[dd, "\epsilon_N"] \arrow[dddl, "\eta"'] \\
 & F_M \arrow[drr, "\widetilde{\varphi}"] \arrow[dl, "*"] & & \\
M \arrow[drr, "\varphi"] & & & N \arrow[dl, "*"] \\
 & & N_* &
\end{tikzcd} \tag{3.4}$$

□

Consider now a morphism $\phi\colon M \to N$ of $\mathfrak{L}_p^{c-1}$-algebras M and N, by identifying N with $(F_N)_*$ we define the *free lift* of ϕ as the morphism $\mathsf{fl}(\phi) := \widetilde{\phi}$ of F_M into F_N given by Proposition 3.1.2. In particular (3.1) holds for ϕ and hence we have

Corollary 3.1.3. *The free lift mapping* fl *is a functor of the category* $\mathfrak{L}_p^{c-1}$ *into* $\mathfrak{L}_p^{c}$*, adjoint to* $*$*. Moreover for any* $\mathfrak{L}_p^{c-1}$*-morphism* $\phi\colon M \to N$*, the square below*

$$\begin{tikzcd}
F_M \arrow[r, "\mathsf{fl}(\phi)"] \arrow[d, "*"] & F_N \arrow[d, "*"] \\
M \arrow[r, "\phi"] & N
\end{tikzcd} \tag{3.5}$$

commutes. Also by (1.15) and the construction of $\mathsf{fl}(\phi)$ *we obtain.*

$$\mathrm{Im}(\mathsf{fl}(\phi)) = \langle \phi(M_1) \rangle^{F_N} \tag{3.6}$$

3.1.1 Isolating essential maximal-weight Relators

With the functor fl it is possible to isolate the *relevant maximal weight relators* in the sense of the following approach.

Consider an object M in $\mathfrak{L}_p^c$ given by $M = \langle M_1 \mid R \rangle$ where R is a usual an homogeneous ideal of $L^c(M_1)$, that is $R = R_2 \oplus \cdots \oplus R_c$. Then M_* may be presented in $\mathfrak{L}_p^{c-1}$ as $L^{c-1}(M_1)/(R_2 + \cdots + R_{c-1})$.

If we present $\mathsf{fl}(M_*) = F_{M_*}$ in $\mathfrak{L}_p^c$ as $\langle M_1 \mid \mathbb{J}(M_*) \rangle$, then by definition $\mathbb{J}(M_*)$ is contained in R.

Denote by $R^c(M)$ the quotient $R/\mathbb{J}(M_*)$, by $\pi_{\mathbb{J}}$ the canonical map modulo $\mathbb{J}(M_*)$ and consider the following morphism of exact sequences. The rightmost square is in $\mathfrak{L}_p^c$.

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\pi_{\mathbb{J}}"] & L^c(M_1) \arrow[r] \arrow[d, "\pi_{\mathbb{J}}"] & M \arrow[d, equal] \\ R^c(M) \arrow[r] & F_{M_*} \arrow[r, "\pi_M"] & M \end{tikzcd} \tag{3.7}$$

where π_M is the natural map of F_{M_*} onto M with kernel $R^c(M)$.

Notice (cfr. Remark 3.1.4 for instance) that $(R^c(M))_i = \mathbf{0}$ for all $2 \leq i < c$. In particular M is $\mathfrak{L}_p^c$-isomorphic to $F_{M_*}/R^c(M)$.

Notice that the map π_M may also be obtained with Proposition 3.1.2 as $\widetilde{\mathrm{id}}_{M_*}$. As such, by (3.1), we have $\pi_M * = *\mathrm{id}_{M_*} \colon F_{M_*} \to M_*$.

In this sense $R^c(M)$ isolates the *essential* relators of M of maximal weight c. Those, which do not arise from relators R_i in lower weight ($i < c$) reaching the weight c by means of Lie brackets.

In Section 3.2, we adopt the second row in diagram (3.7) above as a suitable presentation of M to perform the amalgamation process.

The philosophy behind this definition is the inductive strategy described in the Introduction and will be tested in Section 3.2 below, switching from $\mathfrak{L}_p^2$ to $\mathfrak{L}_p^3$.

3.1.2 Embedding Issues

In the forthcoming sections the functor constructed above will essentially be applied to the following situation: consider an $\mathfrak{L}_p^{c-1}$-subalgebra H of $M \in \mathfrak{L}_p^{c-1}$ and the inclusion $i \colon H \subseteq M$. Denote by γ the lifted morphism $\mathsf{fl}(i)$.

We identify $L^c(H_1)$ with the $\mathfrak{L}_p^c$-subalgebra $\langle H_1 \rangle^{L^c(M_1)}$ of $L^c(M_1)$.

Now consider the $\mathfrak{L}_p^c$-presentations $\langle H_1 \mid \mathbb{J}(H) \rangle$ and $\langle M_1 \mid \mathbb{J}(M) \rangle$ for $F_H = \mathsf{fl}(H)$ and $F_M = \mathsf{fl}(M)$ respectively. Since H is an $\mathfrak{L}_p^{c-1}$-subalgebra of M, we may consider $\mathbb{J}(H)$

as a subspace of $L^c(M_1)$ with $\mathbb{J}(H) \subseteq L^c(H_1) \cap \mathbb{J}(M)$. By Proposition 3.1.2 the map γ coincides with

$$\begin{aligned} \gamma = \mathsf{fl}(i)\colon F_H &\longrightarrow F_M \\ w + \mathbb{J}(H) &\longmapsto w + \mathbb{J}(M) \end{aligned} \tag{3.8}$$

for all w in $L^c(H_1)$.

As a consequence we get

Remark 3.1.4. With the above considerations about notations

$$\ker(\gamma) = (L^c(H_1) \cap \mathbb{J}(M))/\mathbb{J}(H). \tag{3.9}$$

Moreover if M is $L^{c-1}(M_1)/R$ then $H = L^{c-1}(H_1)/R \cap L^{c-1}(H_1)$ and hence for all $i < c$, by (3.3) we have $(L^c(H_1) \cap \mathbb{J}(M))_i = \mathbb{J}(H)_i$. It follows $\ker(\gamma)$ is a homogeneous ideal of total weight c, that is $\ker(\gamma)$ is contained in $(F_H)_c$.

We illustrate below how the free-lift functor actually doesn't preserve – in general – embeddings. This example and the result which follows concern the particular case of lifting $\mathfrak{L}_p^2$-algebras.

Remark 3.1.5. There are extensions of $\mathfrak{L}_p^2$-algebras $M \supseteq H$, such that the map γ of F_H into F_M, defined in (3.8) is not injective.

Proof. Consider an algebra M of $\mathfrak{L}_p^2$ given by the presentation $M = \langle M_1 \mid R^2(M) \rangle$, where M_1 is freely generated by the $\mathbb{Z}_p$-base

$$\mathcal{B} = \{a, u, x, y, z, h_1, \ldots, h_4, e_1, \ldots, e_4\}$$

and $R^2(M)$ is the span in $\bigwedge^2 M_1$ of the following linearly independent relators:

$$\begin{matrix} [h_1, h_2] + [x, a], \\ [h_3, h_4] + [a, u], \end{matrix} \quad \begin{matrix} [z, y] + [u, x], \\ [e_1, e_2] + [y, a], \\ [e_3, e_4] + [a, z]. \end{matrix}$$

Let H_1 denote the subspace of M_1 generated by $\mathcal{B} \backslash \{a\}$ in M_1. Now consider the following homogeneous sum of weight 3 in $L^3(M_1)$:

$$\begin{aligned} &[[h_1, h_2] + \underline{[x, a], u]} + \\ &[[h_3, h_4] + \underline{[a, u], x]} + \\ &\underline{[[z, y], a]} + \underline{[[u, x], a]} + \\ &[[e_1, e_2] + \underline{[y, a], z]} + \\ &[[e_3, e_4] + \underline{[a, z], y]}. \end{aligned}$$

By definition, this lays in $\mathbb{J}(M)$, but after deleting Jacobi sums it is also equal to

$$w := [h_1, h_2, u] + [h_3, h_4, x] + [e_1, e_2, z] + [e_3, e_4, y]$$

and hence belongs to $L^3(H_1)$. By (3.9), now $\bar{w} \in F_H$ is a non-zero element of $\ker(\gamma) = (L^3(H_1) \cap \mathbb{J}(M))/\mathbb{J}(H)$ since $w \notin \mathbb{J}(H)$.

□

Notice that in the example above, H *is not self-sufficient* in M ($\delta(a/H) = -3$). This is actually the only obstruction for extensions of $\widetilde{\mathcal{K}}_2$ not to be lifted to $\mathfrak{L}_p^3$-embeddings by fl:

Proposition 3.1.6. *Let M be a $\widetilde{\mathcal{K}}_2$-algebra. Assume $i\colon H = \langle H_1 \rangle^M \subseteq M$ is a self-sufficient $\mathfrak{L}_p^2$-embedding, then the map $\gamma = \mathsf{fl}(i)\colon F_H \to F_M$ is an $\mathfrak{L}_p^3$-monomorphism.*

By Remark (3.1.4) this is equivalent for a strong extensions $H \leqslant M$ to imply

$$\mathbb{J}(H) = \mathbb{J}(M) \cap L^3(H_1)$$

where as above we let $L^3(H_1)$ coincide with $\langle H_1 \rangle^{L^3(M_1)}$.

Proof. As δ-*strongness* traduces into a local property by Proposition 2.1.14, we may assume without loss of generality, that H_1 is a *finite* subspace of M_1.

If γ denotes $\mathsf{fl}(i)$ as above, assuming γ not injective, yields a nontrivial $w \in (\mathbb{J}(M) \cap L^3(H_1)) \setminus \mathbb{J}(H)$.

As M lays in $\mathfrak{L}_p^2$ and hence $M = L^2(M_1)/R^2(M)$, we have by definition

$$\mathbb{J}(M) = \langle \mu,\ [\nu, z] \mid \mu, \nu \in R^2(M),\ z \in M_1 \rangle_{\mathbb{Z}_p}.$$

Moreover, by the remarks above w may be assumed homogeneous of weight 3, hence a finite sum like

$$w = \sum_\alpha [\nu_\alpha, z_\alpha]. \tag{3.10}$$

for some $\nu_\alpha \in R^2(M)$ and $z_\alpha \in M_1$. On the other hand w must be also identical to a linear combination of monomials of weight 3 in elements of H_1.

Extract a maximal subset $\mathcal{Z}$ out of the z_α's above, which is linearly independent over H_1. With bilinearity of the Lie bracket and since $R^2(M)$ is an additive subgroup of $\bigwedge^2 M_1$, we can actually give expression (3.10) the form

$$w = \sum_{u \in \mathcal{U}_0} [\nu_u, u] + \sum_{z \in \mathcal{Z}} [\nu_z, z] \tag{3.11}$$

where $\mathcal{U}_0$ is a linearly independent subset of H_1, $\mathcal{Z} \subseteq M_1$ is the above set independent over H_1 and ν_u, ν_z belong to $R^2(M)$. (3.11).

Now we claim that sum (3.11) can be transformed into

$$w = \sum_{u \in \mathcal{U}} [\nu_u, u] + \sum_{x \in \mathcal{X}} [\nu_x, x] + \sum_{y \in \mathcal{Y}} [\lambda_y, y] \tag{3.12}$$

where $\mathcal{U}$ is a $\mathbb{Z}_p$-independent subset of H_1, $\mathcal{X}\mathcal{Y}$ is linearly independent over H_1 in M_1, the set $\{\nu_u, \nu_x \mid u \in \mathcal{U}, x \in \mathcal{X}\}$ is linearly independent over $\bigwedge^2 H_1$ in $R^2(M)$, and $\{\lambda_y \mid y \in \mathcal{Y}\}$ is an independent subset of $R^2(H)$.

To obtain (3.12) from (3.11) we adopt the following steps. Let first $\mathcal{X}$ be a maximal subset of $\mathcal{Z}$ with the property that $\{\nu_x \mid x \in \mathcal{X}\}$ is an $\bigwedge^2 H_1$-independent subset of the ν_z's. We transform by bilinearity of the Lie product, the sum $\sum_{z\in\mathcal{Z}}[\nu_z, z]$ of (3.11) into

$$\sum_{x\in\mathcal{X}}[\nu_x, x] + \sum_{y\in\mathcal{Y}}[\lambda_y, y]$$

where $\mathcal{Y}$ is a subset of $\mathcal{Z}$ such that $\mathcal{X}\mathcal{Y}$ and the λ_y's have the properties claimed for (3.12).

To get (3.12) we now modify the set $\{\nu_u, \nu_x \mid u \in \mathcal{U}_0, x \in \mathcal{X}\}$ to get an $\bigwedge^2 H_1$-independent one, this will possibly reduce the length $|\mathcal{U}_0|$ of the first segment of (3.11).

Assume $\mathcal{U}$ is a maximal subset of $\mathcal{U}_0$ for which $\{\nu_u \mid u \in \mathcal{U}\}$ is linearly independent over $\langle \bigwedge^2 H_1, \nu_x \mid x \in \mathcal{X}\rangle_{\mathbb{Z}_p}$ in $\bigwedge^2 M_1$. Then for any $u_0 \in \mathcal{U}_0 \setminus \mathcal{U}$ we have

$$\nu_{u_0} = \sum_{u\in\mathcal{U}} t_u \nu_u + \sum_{x\in\mathcal{X}} s_x \nu_x + \eta$$

where t_u and s_u are in $\mathbb{Z}_p$ for all u and η is an element of $R^2(H)$. This yields

$$[\nu_{u_0}, u_0] = \sum_{u\in\mathcal{U}}[\nu_u, t_u u_0] + \sum_{x\in\mathcal{X}}[\nu_x, s_x u_0] + [\eta, u_0]. \tag{3.13}$$

Since w is not in $\mathbb{J}(H)$, by replacing w with $w - [\eta, u_0]$, we still obtain an element of $L^3(H_1) \cap \mathbb{J}(M)$ which does not belong to $\mathbb{J}(H)$. On the other hand we merge[2] the remaining terms of (3.13) into

$$w = \sum_{u\in\mathcal{U}_0\setminus\mathcal{U}, u_0}[\nu_u, u] + \sum_{u\in\mathcal{U}}[\nu_u, u + t_u u_0] + \sum_{x\in\mathcal{X}}[\nu_x, x + s_x u_0] + \sum_{y\in\mathcal{Y}}[\lambda_y, y]$$

Now the sets

$$\{u + t_u u_0 \mid u \in \mathcal{U}\} \quad \text{and} \quad \{x + s_x u_0 \mid x \in \mathcal{X}\}$$

are again respectively linearly independent and linearly independent over $\langle \mathcal{Y}, H_1\rangle_{\mathbb{Z}_p}$.

Iterating this step for all $u \in \mathcal{U}_0 \setminus \mathcal{U}$ – each time renaming the u's, the x's and w – we reach the desired expression (3.12). At the end (3.12) is not trivial if w doesn't lay in $\mathbb{J}(H)$.

Now arrange the above sets into a linearly ordered base $\mathcal{M}$ of M_1 according to the following hierarchy:

$$\mathcal{M} = \{\mathcal{U} > \mathcal{H} > \mathcal{X} > \mathcal{Y} > \mathcal{W}\}$$

where $\mathcal{H}$ is a completion of $\mathcal{U}$ to a base of H_1, $\mathcal{W}$ is a completion of $\mathcal{U}\mathcal{H}\mathcal{X}\mathcal{Y}$ to a base of M_1 and each of the above subparts of $\mathcal{M}$ is ordered in some way.

[2] This is actually how we reached expressions (3.11) and (3.12).

According to Definitions 1.4.4 and 1.4.11 and Fact 1.4.10, we write elements ν_u, ν_x and λ_y in (3.12) as $\mathbb{Z}_p$-linear combinations of basic $\mathcal{M}$-monomials of weight 2.

As a result, for suitable scalars $\theta_\alpha \in \mathbb{Z}_p$, the sum (3.12) becomes a linear combination

$$\sum_\alpha \theta_\alpha [a_\alpha, b_\alpha, z_\alpha] \tag{3.14}$$

of left-normed commutators of weight 3, where each monomial $[a_\alpha, b_\alpha, z_\alpha] = [a, b, z]$ has a, b laying in $\mathcal{M}$ with $a > b$ while z belongs to $\mathcal{UXY}$. If in addition $z \geq b$, the term $[a, b, z]$ is a basic monomial of weight 3. If on the contrary $a > b > z$, then we call the monomial $[a, b, z]$ a *prebasic* monomial.

Applying the Jacobi Identity, every prebasic monomial $[a, b, z]$ can be transformed (cfr. [Hal50, p.577]) in the sum of two basic commutators, namely

$$[a, b, z] = [a, z, b] - [b, z, a]. \tag{3.15}$$

On the other hand, with Fact 1.4.10 again, as an element of $L^3(H_1)$, w admits a unique expression as a linear combination $\mathcal{B}^H$ of basic monomials over $\mathcal{UH}$ of weight 3.

We have then

$$\mathcal{B}^H = w = \mathcal{B} + p\mathcal{B} = \mathcal{B} + \mathcal{B}_* \tag{3.16}$$

where $\mathcal{B}$, $p\mathcal{B}$ are sums of respectively basic and prebasic commutators over $\mathcal{M}$ representing (3.14) and $\mathcal{B}_*$ is the sum of basic $\mathcal{M}$-monomials arising from $p\mathcal{B}$ by means of substitutions (3.15).

By abuse of notation, we let $\mathcal{B}$, $\mathcal{B}^H$, $p\mathcal{B}$ and $\mathcal{B}_*$ also denote the *sets* of monomials which appear in the corresponding sum.

From a comparison of equality $\mathcal{B}^H = \mathcal{B} + \mathcal{B}_*$ and by unicity in Fact 1.4.10, it follows that $\mathcal{B}^H \subseteq \mathcal{B}\mathcal{B}_*$ and each basic $\mathcal{M}$-monomial in $\mathcal{B}\mathcal{B}_*$ which is not in $\mathcal{B}^H$, must be cancelled from the sum $\mathcal{B} + \mathcal{B}_*$ by the same commutator with opposite coefficient, the latter laying again in $\mathcal{B}\mathcal{B}_*$. This happens in particular of all basic terms containing elements of $\mathcal{M}$ which are not in H_1.

Assume a term $[a, b, z]$ appearing in (3.14) as a $\mathcal{B}$-element is to be cancelled, then the same commutator, with opposite sign will be necessarily found in $\mathcal{B}_*$ and not of course in $\mathcal{B}$ again. The same holds with the roles of $\mathcal{B}$ and $\mathcal{B}_*$ exchanged.

Also notice that the elements of $\mathcal{M}$, appearing in the rightmost entry of the Lie brackets in (3.14) force the sum to be grouped after the labels $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{Y}$.

(Claim 1) Monomials $[a, b, z]$ appearing in (3.14) do not have entries from $\mathcal{W}$.

Assume on the contrary, (3.14) contains a commutator $[a, v, z]$ with $v \in \mathcal{W}$, $z \in \mathcal{UX}$. Then necessarily $z \geq v$ and $[a, v, z]$ is basic. It follows that, monomials whose support meets $\mathcal{W}$ cannot appear in $p\mathcal{B}$ and then $\mathcal{B}_*$-basic terms will not contain $\mathcal{W}$-elements. By the above remarks, we conclude, there is no hope for $[a, v, z]$ to be cancelled from $\mathcal{B}$ and this implies such terms simply don't occur.

In particular, all ν_u and ν_x have support contained in $\mathcal{UHXY}$.

(Claim 2) The sum (3.12) contains terms $[\nu_*, *]$ with $* \in \mathcal{XY}$. Concisely $\mathcal{XY} \neq \varnothing$. Moreover $\mathcal{U}$ cannot be empty either.

Assume $w = \sum_{u \in \mathcal{U}} [\nu_u, u]$ only. As the ν_u's are independent over $\bigwedge^2 H_1$, then there is at least one monomial $[a, b, u]$ of (3.14) with $[a, b]$ not entirely supported on $\mathcal{UH}$. This would contradict (Claim 1).

We also have $\mathcal{U} \neq \varnothing$, for if in $\mathcal{B} + \mathcal{B}_*$ every monomial contains an element from $\mathcal{XY}$, the entire expression would cancel although w is nontrivial.

(Claim 3) The support of each λ_y is contained in $\mathcal{U}$.

Assume not. Then in (3.14) appears a term $[a, b, y]$ with a or b in $\mathcal{H}$. As $a > b$ and $\mathcal{U} > \mathcal{H} > \mathcal{Y}$, it follows necessarily $b \in \mathcal{H}$ and $[a, b, y]$ is prebasic, its transformation in two $\mathcal{B}_*$-elements produces basic terms $[a, y, b]$ and $[b, y, a]$, both not in $\mathcal{B}^H$. On the other hand, the commutator $[a, y, b]$ cannot be found in part $\mathcal{B}$ of (3.16) and will not be cancelled. This is a contradiction

We eventually prove the assertion of the lemma contradicting the self-sufficiency of H in M.

Consider the subspace $C_1 = \langle H_1, \mathcal{X}, \mathcal{Y} \rangle_{\mathbb{Z}_p}$ of M_1. On one hand by (Claim 2) $\mathcal{U} \neq \varnothing$ and $C \supsetneq H$, while (Claim 3) together with axiom $\Sigma^2(2)$ imply $|\mathcal{Y}| < |\mathcal{U}|$ as $\delta(\mathcal{U})$ must be positive and the λ_y's are in $\bigwedge^2 \langle \mathcal{U} \rangle_{\mathbb{Z}_p}$.

On the other hand by (Claim 1), since the ν_u, ν_x's are relators in $R^2(M)$, which are linearly independent over $\bigwedge^2 H_1$ and have support inside C_1, we have $\delta(C/H) = \dim_{\mathbb{Z}_p}(C_1/H_1) - \dim_{\mathbb{Z}_p}(R^2(C/H)) \leq |\mathcal{X}| + |\mathcal{Y}| - (|\mathcal{X}| + |\mathcal{U}|) < 0$ which is impossible. The proof is now complete.

□

3.2 Predimensions for the third nilpotent Class

For the rest of the chapter we assume a prime p has been fixed, greater than 3. We will consider Lie algebras of $\mathfrak{L}_p^3$.

For a chosen M in $\mathfrak{L}_p^3$ and any $\mathfrak{L}_p^3$-subalgebra A of M, the truncation A_* of Section 3.1, is $\mathfrak{L}_p^2$-isomorphic to $\langle A_1 \rangle^{M_*}$. These algebras will be identified in the sequel.

This means, we can apply on M_1 the "pregeometric machinery" introduced in Chapter 2, associated to M_*. With this purpose we rename by δ_2, the nil-2 deficiency δ of Definition 2.1.1 and set, for finite subspaces A_1 of M_1

$$\delta_2(A_1) = \dim_{\mathbb{Z}_p}(A_1) - R^2(A_*).$$

Following our previous convention, the same integer will be equal to $\delta_2(A)$ for ease of notation.

On the same line, for any $\mathfrak{L}_p^3$-extension M of N, we will write $N \leqslant_2 M$ if N_* is self-sufficient in M_* according to Definition 2.1.6. The same meaning is attributed to the expression $N_1 \leqslant_2 M_1$.

Moreover, for a given M in $\mathfrak{L}_p^3$, we denote by d_2^M the dimension function on M_1 obtained by the predimension δ_2.

On the other hand for an $\mathfrak{L}_p^3$-algebra A, we have from (3.7) of Section 3.1.1 above, the following *lifted* presentation:

$$R^3(A) \longrightarrow F_{A_*} \xrightarrow{\pi_A} A \tag{3.17}$$

and in particular

$$A \simeq_{\mathfrak{L}_p^3} \frac{F_{A_*}}{R^3(A)} = A_* \oplus \frac{(F_{A_*})_3}{R^3(A)}.$$

This yields a new integer invariant attached to $\mathfrak{L}_p^3$-objects, defined in the following

Definition 3.2.1. For a finitely generated $\mathfrak{L}_p^3$-algebra A we define the $\mathfrak{L}_p^3$-deficiency as the integer

$$\delta_3(A) = \delta_2(A) - \dim_{\mathbb{Z}_p}\Big(R^3(A)\Big). \tag{3.18}$$

In particular $\delta_3(A)$ depends only of the quantifier-free $\mathscr{L}^3$-diagram of A. As a consequence a possible lower bound to δ_3 in M_1 is axiomatisable via $\mathscr{L}^3$-sentences.

In the scope of section 1.4.1, if we compare (3.18) above with (1.28) on page 33, we obtain the same thing, just differently organised. In fact if we consider (1.27) we get

$$\dim_{\mathbb{Z}_p}(H_2(A, \mathfrak{L}_p^3)) = \dim_{\mathbb{Z}_p}(R^2(A_*)) + \dim_{\mathbb{Z}_p}(R^3(A)).$$

This said, in view of a definition of self-sufficiency for $\mathfrak{L}_p^3$-extensions, we need a different notion of predimension, which emulates the local property (2.1) of R^2 and ease computations for a future notion of free $\mathfrak{L}_p^3$-amalgam.

The point here is that for an arbitrary $A \subseteq M$ like above, $R^3(A)$ is not in general a subspace of $R^3(M)$.

Take an $\mathfrak{L}_p^3$-inclusion $i\colon H \subseteq M$, this means as usual $H = \langle H_1 \rangle^M$ for $H_1 \subseteq M_1$ and consider the truncation to $\mathfrak{L}_p^2$, $i_*\colon H_* \subseteq M_*$. We denote by γ_H^M the map described in (3.8)

$$\gamma_H^M := \mathsf{fl}(i_*)\colon F_{H_*} \longrightarrow F_{M_*}. \tag{3.19}$$

By (3.6) and (3.9), we have

$$\operatorname{im}(\gamma_H^M) = \langle H_1 \rangle^{F_{M_*}} \quad \text{and} \quad \ker(\gamma_H^M) = \frac{L^3(H_1) \cap \mathbb{J}(M_*)}{\mathbb{J}(H_*)} \tag{3.20}$$

while by Proposition 3.1.6 we obtain for all $\mathfrak{L}_p^3$-subalgebras $H \subseteq M$ as above

Corollary 3.2.2. *If $H \leqslant_2 M$ then $\gamma_H^M\colon F_{H_*} \to F_{M_*}$ is injective.*

We want also, the new presentation obtained in (3.17) to interact with subalgebras, that is

Lemma 3.2.3. *For an $\mathfrak{L}_p^3$-subalgebra H of M, we obtain the following commutative diagram with exact rows.*

$$
\begin{tikzcd}
R^3(H) \arrow[r] \arrow[d, "\gamma_H^M"] & F_{H_*} \arrow[r, "\pi_H"] \arrow[d, "\gamma_H^M"] & H \arrow[d, "i"] \\
R^3(M) \arrow[r] & F_{M_*} \arrow[r, "\pi_M"] & M
\end{tikzcd}
\tag{3.21}
$$

In particular $\ker(\gamma_H^M) \subseteq R^3(H)$.

Proof. We show that the rightmost square in (3.21) commutes. This follows by proposition 3.1.2 and the fact $(\gamma_H^M \pi_M)* = * \, i_* = (\pi_H i)*$ applied to the diagrams

$$
\begin{tikzcd}
F_{H_*} \arrow[r, shift left, "\gamma_H^M \pi_M"] \arrow[r, shift right, "\pi_H i"'] \arrow[d, "*"] & M \arrow[d, "*"] \\
H_* \arrow[r, "i_*"'] & M_*
\end{tikzcd}
\tag{3.22}
$$

□

This allows us to define a more *adaptive* deficiency, which depends of the embedding in the ambient structure M.

Definition 3.2.4. Let M be an $\mathfrak{L}_p^3$-algebra. For any $H_1 \subseteq M_1$. We set

$$R_M^3(H) = R_M^3(H_1) = \gamma_H^M \left(R^3(H) \right)$$

and define for finitely generated H

$$\partial_3^M(H) = d_2^M(H_1) - \dim_{\mathbb{Z}_p}(R_M^3(H)) \tag{3.23}$$

and for any $\mathfrak{L}_p^3$-subalgebra N of M and finite $H_1 \subseteq M_1$

$$\partial_3^M(H/N) = d_2^M(H/N) - \dim_{\mathbb{Z}_p}(R_M^3(H/N)) \tag{3.24}$$

where $R_M^3(H/N)$ is the quotient $\mathbb{Z}_p$-vector space $R_M^3(N+H)/R_M^3(N)$. As before for $N + H$ is meant $\langle N_1 + H_1 \rangle^M$ and in general any of the expressions above are allowed to carry indices $_1$. In fact as in the nil-2 case, we are searching for a notion of predimension – and eventually a pregeometry – which is concerned with sets from the *sort* M_1.

Remark 3.2.5. By (3.20) and (3.21) we have

1. for $H_1 \subseteq N_1 \subseteq M_1$ we have

$$R^3_M(H) = R^3(M) \cap \operatorname{im}(\gamma^M_H) = R^3(M) \cap \langle H_1 \rangle^{F_M} \tag{3.25}$$

2. since $\gamma^N_H \gamma^M_N = \gamma^M_H$, $R^3_N(H_1)$ maps *onto* $R^3_M(H_1)$ via γ^M_N.

In particular we obtain a form of the local relators, which is a lot similar to $R^2_M(H)$ in (2.1) of Chapter 2.

Assume A is a finite $\mathfrak{L}^3_p$-subalgebra of M with $A \leqslant_2 M$, by Corollary 3.2.2 above follows, that $R^3(A) \simeq_{\mathbb{Z}_p} R^3_M(A)$ and hence, as $d^M_2(A) = \delta_2(A)$, we have $\delta_3(A) = \partial^M_3(A)$. In particular, by Lemma 2.1.15 and Remark 3.2.5.(2.) we have.

Lemma 3.2.6. *For a given $\mathfrak{L}^3_p$-algebra M, the integers $\partial^M_3(A)$ and $\delta_3(A)$ do coincide on all finitely generated $\mathfrak{L}^3_p$-subalgebras A of M when A_* is self-sufficient in M_* with respect to δ_2.*

Moreover ∂^M_3 coincides with ∂^N_3 on the subspaces of N_1, for all δ_2-strong extensions $N \leqslant_2 M$.

In Section 3.2.2 below we actually show that δ_3 and ∂_3 are always comparable, in the direction $\partial^M_3 \leq \delta_3$ for all M.

It is worth to mention here, that for a given M and subspaces H_1, K_1 of M_1, it is not in general the case that

$$\langle H_1 \rangle^{F_M} \cap \langle K_1 \rangle^{F_M} \quad \text{and} \quad \langle H_1 \cap K_1 \rangle^{F_M} \tag{3.26}$$

coincide.

It is also not true in general that $R^3_M(H_1) \cap R^3_M(K)$ equals $R^3_M(H_1 \cap K_1)$ and the analogous of *submodularity* (2.5) of Section 2.1 for δ_3 and ∂_3 may fail. In fact we have

Remark 3.2.7. ∂^M_3 and δ_3 are not in general submodular.

Proof. Consider the $\mathfrak{L}^3_p$-algebra given by the presentation $M = \langle M_1 \mid R \rangle$, where M_1 is the vector space over $\mathbb{Z}_p$ with basis $\{a, b_1, \dots, b_4, m_1, m_2\}$ and R is the ideal of $L^3(M_1)$ generated by the relators

$$\rho = [a, b_1] - [b_2, b_3] \qquad \begin{aligned} \alpha &= [b_2, b_3, b_4] - [m_1, m_2, m_2], \\ \beta &= [b_2, b_3, m_1] - [b_4, m_2, m_2]. \end{aligned} \tag{3.27}$$

In this algebra, $\mathbb{J}(M)$ is the ideal of $L^3(M_1)$ generated by ρ.

Now denote with N_1 the subspace of M_1 generated by $b_1, \dots, b_4, m_1, m_2$ and let E_1 be $\langle a, b_1, b_4, m_1, m_2 \rangle_{\mathbb{Z}_p}$. We have $\alpha \equiv [a, b_1, b_4] - [m_1, m_2, m_2]$ and $\beta \equiv [a, b_1, m_1] - [b_4, m_2, m_2]$ modulo $\mathbb{J}(M)$, and hence $\bar{\alpha} \in R^3_M(E_1) \setminus R^3_M(E_1 \cap N_1)$.

As $\langle N_1, a \rangle = M$, we have $d^M_2(E/N) = d^M_2(a/N) = \delta_2(a/N) = 0$ while $d^M_2(E_1/E_1 \cap N_1) = \delta_2(E_1/E_1 \cap N_1) = 1$. This means

$$0 = \partial^M_3(E/N) > \partial^M_3(E_1/E_1 \cap N_1) = -1$$

and, as $E, N \leqslant_2 M$, $\delta_3(E_1 + N_1) + \delta_3(E_1 \cap N_1) > \delta_3(E) + \delta_3(N)$.

□

We now define self-sufficiency on extensions of $\mathfrak{L}_p^3$-algebras.

Definition 3.2.8. We say that an $\mathfrak{L}_p^3$-subalgebra H of M is self-sufficient and write $H \leqslant_3 M$ if

- $H \leqslant_2 M$ and
- $\partial_3^M(E/H) \geq 0$ for all finite subspaces $E_1 \subseteq M_1$.

By the first condition, and the following Lemma 3.2.12, it is possible to express $H \leqslant_3 M$ in terms of δ_3 as well[3]. For a finite H – say – this property is actually part of the elementary type of H.

3.2.1 A first asymmetric Amalgamation

We now describe a free amalgamation construction for $\mathfrak{L}_p^3$ algebras. We want to proceed as similar as possible to amalgamation in $\mathfrak{L}_p^2$ and prove the analogous of the *asimmetric* Lemma 2.2.7. As we have seen in Section 2.3, the asymmetric amalgam allows the approximation of richness in the axiomatisation.

We first prove a result concerning the modular issue (3.26), we show that under *free composition*, the intersection of subalgebras is actually preserved under the free lift.

Switch back to nil-2 algebras for a moment and take $M \in \mathfrak{L}_p^2$ and subspaces H_1, K_1 of M_1. Since $\bigwedge^2 H_1 \cap \bigwedge^2 K_1 = \bigwedge^2 H_1 \cap K_1$, then it is straightforward to verify, that the condition

$$\langle H_1 \rangle^M \cap \langle K_1 \rangle^M = \langle H_1 \cap K_1 \rangle^M \tag{3.28}$$

is equivalent to require

$$R^2(M) \cap \left(\bigwedge^2 H_1 + \bigwedge^2 K_1 \right) = R^2(H) + R^2(K).$$

and that the last equality is satisfied in particular when H and K are in free composition in M (cfr. Definition 2.2.3).

In fact free amalgams imply that condition (3.28) remains true of the free-lifted algebras:

Lemma 3.2.9. *Assume M is the free amalgam $N \circledast_B A$ of $\mathfrak{L}_p^2$-algebras $N \supseteq B \subseteq A$. We identify as usual $L^3(N_1)$ and $L^3(A_1)$ with subalgebras of $L^3(M_1)$. Then we have*

$$\mathbb{J}(M) \cap (L^3(N_1) + L^3(A_1)) = \mathbb{J}(N) + \mathbb{J}(A). \tag{3.29}$$

[3] One defines of course $\delta_3(E/H)$ as $\delta_2(E/H) - \dim_{\mathbb{Z}_p}(R^3(H+E)/R^3(H))$.

and also

$$\langle N_1\rangle^{F_M} \cap \langle A_1\rangle^{F_M} = \langle N_1 \cap A_1\rangle^{F_M} = \langle B_1\rangle^{F_M}. \tag{3.30}$$

Proof. Since $L^3(A_1) \cap L^3(N_1) = L^3(A_1 \cap N_1)$, the second statement follows easily from the first, considering the above arguments for nil-2 algebras.

To prove (3.30) assume that $w_N + w_A \in \mathbb{J}(M)$ for some w_N and w_A in $L^3(N_1)$ and $L^3(A_1)$ respectively. By (3.28), we may assume, these are homogeneous elements of weight 3.

We arrange a basis X for M_1 as follows $X = \{X^a > X^B > X^n\}$, where X^B is a basis for B_1, $X^B X^n$ is a basis for N_1 and $X^a X^B$ is a basis for A_1.

Now since $R^2(M) = R^2(N) + R^2(A)$, we have

$$\mathbb{J}(M)_3 = \langle \mathbb{J}(A), \mathbb{J}(N), [R^2(A), N_1], [R^2(N), A_1]\rangle_{\mathbb{Z}_p}$$

and without loss of generality, $w_N + w_A$ may be written as a sum of terms like $[\nu_N, x]$ and $[\nu_A, y]$ with $x \in X^a$, $y \in X^n$ and $\nu_A \in R^2(A)$ and $\nu_N \in R^2(N)$.

We proceed with the terminology used in the proof of Proposition 3.1.6 and obtain – after each ν_A and ν_N has been expressed as sums of basic X-monomials of weight 2 – an equality in $L^3(M_1)$:

$$w_N + w_A = \mathcal{B}^{N,a} + p\mathcal{B}^{A,m}$$

where

$\mathcal{B}^{N,a}$ is a sum of basic terms $[m_1, m_2, x]$ for $m_i \in X^B X^n$

$p\mathcal{B}^{A,m}$ is a sum of prebasic terms $[a_1, a_2, y]$ for $a_i \in X^a X^B$

and x, y are as required above.

We now transform all prebasic monomials above into $[a_1, y, a_2] - [a_2, y, a_1]$ which are basic and whose sum we denote by $\mathcal{B}^{A,m}_*$.

We obtain a sum of basic commutators

$$w_N + w_A = \mathcal{B}^{N,a} + \mathcal{B}^{A,m}_*.$$

On the other hand the unique basic expression from Fact 1.4.10 for $w_N + w_A$ does not involve *mixed* terms, that is monomials whose support meets both X^a and X^n non-trivially.

As a consequence, all mixed terms must cancel each other from the sum $\mathcal{B}^{N,a} + \mathcal{B}^{A,m}_*$ and cancellations do not arise within the same group. The only possibility instead, is that mixed $\mathcal{B}^{A,m}_*$-monomials cancel mixed $\mathcal{B}^{N,a}$-monomials and vice versa.

Consider indeed a term $[m_1, m_2, x]$ above with – say – $m_2 \in X^n$, this is to be neutralised by the prebasic commutator $[x, m_1, m_2]$, which has to lay in $p\mathcal{B}^{A,m}$. This yields $m_1 \in X^B$ and implies $\mathcal{B}^{A,m}_*$ contains the basic commutator $[x, m_2, m_1]$ which differs from any $\mathcal{B}^{N,a}$-term. We deduce that no mixed $\mathcal{B}^{N,a}$-term is present in the sum above, and, with much similar arguments no mixed $p\mathcal{B}^{A,m}$-term shows up as well. This means that all monomials $[m_1, m_2]$ and $[a_1, a_2]$ above must belong to $\bigwedge^2 B_1$. Thus both ν_A and ν_N belong to $R^2(B)$ and the assertion follows.

$\square$

As the plan is to consider an asymmetric configuration, we start from $\mathfrak{L}_p^3$-algebras $N \supseteq B \leqslant_2 A$ and let M_* *denote* the $\mathfrak{L}_p^2$-free amalgam $N_* \circledast_{B_*} A_*$.

Now by Lemma 2.2.7 follows $N_* \leqslant M_* \supseteq A_*$ and hence both $\gamma_B^A \colon F_{B_*} \to F_{A_*}$ and $\gamma_N^M := \mathsf{fl}(i : N_* \subseteq M_*) \colon F_{N_*} \to F_{M_*}$ are monomorphisms.

If we set $K_B = \ker(\gamma_B^N)$ and $K_A = \ker(\gamma_A^M)$, then since $\gamma_B^N \gamma_N^M = \gamma_B^A \gamma_A^M$, we have by (3.20)

$$\gamma_B^A(K_B) = K_A \cap \langle B_1 \rangle^{F_A}.$$

On the other hand by (3.29) and (3.20) follows that

$$K_A = \frac{L^3(A_1) \cap \mathbb{J}(M)}{\mathbb{J}(A)} = \frac{(L^3(A_1) \cap \mathbb{J}(N)) + \mathbb{J}(A)}{\mathbb{J}(A)} \subseteq \langle B_1 \rangle^{F_A}$$

and hence $\gamma_B^A(K_B) = K_A$.

If now $\bar{\gamma}_B^N$, $\bar{\gamma}_B^A$ and $\bar{\gamma}_A^M$ denote the quotient maps modulo K_B and K_A respectively, we obtain the following *injective* commutative arrows:

$$\begin{tikzcd}
 & F_{M_*} & \\
F_{N_*} \arrow[ur, "\gamma_N^M"] & & F_{A_*}/K_A \arrow[ul, "\bar{\gamma}_A^M"'] \\
 & F_{B_*}/K_B \arrow[ul, "\bar{\gamma}_B^N"] \arrow[ur, "\bar{\gamma}_B^A"'] &
\end{tikzcd} \tag{3.31}$$

We also have $\langle N_1 \rangle^{F_{M_*}} = \gamma_N^M(F_{N_*})$ and $\langle A_1 \rangle^{F_{M_*}} = \gamma_A^M(F_{A_*}) = \bar{\gamma}_A^M(F_{A_*}/K_A)$.

Furthermore, by Lemma 3.2.3 we have $K_A \subseteq R^3(A)$ and $K_B \subseteq R^3(B)$. We can therefore rebuild A and B as quotients

$$A \simeq \frac{F_{A_*}}{R^3(A)} \simeq \frac{F_{A_*}/K_A}{R^3(A)/K_A} \text{ and similarly } B \simeq \frac{F_{B_*}/K_B}{R^3(B)/K_B}. \tag{3.32}$$

Now by Lemma 3.2.9 we get

$$\langle N_1 \rangle^{F_{M_*}} \cap \langle A_1 \rangle^{F_{M_*}} = \langle N_1 \cap A_1 \rangle^{F_{M_*}} = \langle B_1 \rangle^{F_{M_*}}. \tag{3.33}$$

Define $R_A := \gamma_A^M(R^3(A)) = \bar{\gamma}_A^M(R^3(A)/K_A)$ and $R_N := \gamma_N^M(R^3(N)) \simeq R^3(N)$ and set

$$\begin{aligned} R_B := \gamma_A^M(R_A^3(B_1)) = R_A \cap \langle B_1\rangle^{F_{M_*}} \\ = \bar{\gamma}_A^M(\bar{\gamma}_B^A(R^3(B)/K_B)) = \gamma_N^M(\bar{\gamma}_B^N(R^3(B)/K_B)) = \\ = \gamma_N^M(R_N^3(B_1)) = R_N \cap \langle B_1\rangle^{F_{M_*}}. \end{aligned} \quad (3.34)$$

Now as $(R^3(N))_2 = \mathbf{0} = (R^3(A))_2$, both R_N and R_A are homogeneous subspaces of F_{M_*} of weight 3 ($R_N, R_A \subseteq (F_{M_*})_3$) and in particular ideals of F_{M_*}. This allows us to define an algebra of $\mathfrak{L}_p^3$

$$M := \frac{F_{M_*}}{R_N + R_A}. \quad (3.35)$$

Now we see M/M_3 coincides *a posteriori* with $M_* = N_* \circledast_{B_*} A_*$ constructed above and hence $R^3(M)$ equals $R_N + R_A$. In particular by (3.33) and (3.34) we obtain

$$R_M^3(A) = R^3(M) \cap \langle A_1\rangle^{F_{M_*}} = R_A \quad (3.36)$$

$$R_M^3(N) = R^3(M) \cap \langle N_1\rangle^{F_{M_*}} = R_N \quad (3.37)$$

and therefore both A and N embeds into M as $\mathfrak{L}_p^3$-subalgebras. Moreover (3.34) and (3.36) or (3.37) give

$$R_M^3(B) = R^3(M) \cap \langle B_1\rangle^{F_{M_*}} = R_B. \quad (3.38)$$

This means M *amalgamates* N *and* A *over* B in $\mathfrak{L}_p^3$ (rewrite Definition 2.2.1 for $\mathfrak{L}_p^3$). In particular with the above defined structures we can now show the following.

Lemma 3.2.10. *Let $N \supseteq B \leqslant_3 A$ be $\mathfrak{L}_p^3$-extensions and assume M is the $\mathfrak{L}_p^3$-algebra defined in (3.35), then $N \leqslant_3 M \supseteq A$.*

Proof. As $N \leqslant_2 M$ by construction, we have to show that for any finite subspace E_1 of M_1, we have $\partial_3^M(E/N) \geq 0$.

Since by 2.1.17, $d_2(E_1/N_1) = d_2(ssc(N_1 + E_1)/N_1)$ and hence

$$\partial_3^M(E/N) \geq \partial_3^M(ssc(N_1 + E_1)/N).$$

It is then sufficient to test $\partial_3^M(E/N)$ on δ_2-self-sufficient subspaces $E_1 \leqslant_2 M_1$ containing N_1 and of finite dimension over N_1.

By Corollary 2.2.8, since $E_1 \supseteq N_1$ we have that M_* is also free amalgam of $\langle E_1\rangle^{M_*}$ and A_* over $\langle E_1 \cap A_1\rangle^{M_*}$. By (3.30) of Lemma 3.2.9 we have therefore

$$\langle E_1\rangle^{F_{M_*}} \cap \langle A_1\rangle^{F_{M_*}} = \langle E_1 \cap A_1\rangle^{F_{M_*}}. \quad (3.39)$$

This equality, with (3.36) and (3.37) imply

$$\begin{aligned} R^3_M(E_1) &= (R_N + R_A) \cap \langle E_1 \rangle^{F_{M_*}} = R_N + (R_A \cap \langle E_1 \rangle^{F_{M_*}}) = \\ &= R_N + (R^3(M) \cap \langle E_1 \cap A_1 \rangle^{F_{M_*}}) = R^3_M(N) + R^3_M(E_1 \cap A_1). \end{aligned}$$

Therefore by (3.38) and (3.39)

$$R^3_M(E)/R^3_M(N) \simeq_{\mathbb{Z}_p} R^3_M(E_1 \cap A_1)/R^3_M(E_1 \cap A_1) \cap R^3_M(N) = R^3_M(E_1 \cap A_1)/R^3_M(B)$$

which is the image of $R^3_A(E_1 \cap A_1)/R^3_A(B)$ through γ^M_A by Remark 3.2.5.

On the other hand by Proposition 2.1.17, Lemma 2.1.12 and by identity (2.8) on page 45 we have $d^M_2(E/N) = \delta(E/N) = \delta(E_1 \cap A_1/B) = d^A_2(E_1 \cap A_1/B)$.

In the end we have $\partial^M_3(E/N) \geq \partial^A_3(E_1 \cap A_1/B_1) \geq 0$ since $B \leqslant_3 A$.

□

The above result has been proved in a non-symmetric fashion. As we have seen in the axiomatisation of T^2, this will be used in a possible first-order approximation of richness in terms of $\mathscr{L}^3$-formulas, should a Fraïssé model be constructed inside $\mathfrak{L}^3_p$.

Of course a symmetric statement holds as well:

Corollary 3.2.11. *Given strong $\mathfrak{L}^3_p$-extensions $N\ {}_3\!\geqslant B \leqslant_3 A$, it is possible to find $M \in \mathfrak{L}^3_p$ with $N \leqslant_3 M\ {}_3\!\geqslant A$, which amalgamates N and A over B.*

M is isomorphic (with loose notation) to

$$\frac{\mathsf{fl}(N_* \circledast_{B_*} A_*)}{R^3(N) + R^3(A)}$$

Notice that the example constructed in Remark 3.2.7 employes an algebra $M = \langle m_1, m_2, B_1, a \mid \rho, \alpha, \beta \rangle$ which is obtained by the underlying free amalgam $M_* = N_* \circledast_{B_*} A_*$ as in (3.35) by taking $B_1 = \langle b_1, \ldots, b_4 \rangle_{\mathbb{Z}_p}$ and $A_1 = \langle B_1, a \rangle_{\mathbb{Z}_p}$.

In M submodularity of δ_3 fails because $\langle E_1 \cap N_1 \rangle^{F_{M_*}} \subsetneq \langle E_1 \rangle^{F_{M_*}} \cap \langle N_1 \rangle^{F_{M_*}}$ but in this case we have $\langle E_1 \cap N_1 \rangle^{M_*} \subsetneq \langle E_1 \rangle^{M_*} \cap \langle N_1 \rangle^{M_*}$ as well.

It is possible to build similar examples, in which such a modular behaviour is true of the $\mathfrak{L}^2_p$-truncated algebra, but not in the free-lifted $\mathfrak{L}^3_p$-structure.

As (3.30) does not hold in general, then in particular $R^3(\ldots)$ fails to be modular. As a consequence we cannot easily decide whether $\leqslant_3$ is transitive (cfr. Lemma 2.1.11). As pointed out in the introduction, we cannot adopt the solution of redefining self-sufficiency by requiring for instance $A \leqslant_3 B$ whenever $\delta_3(X \cap A_1) \leq \delta_3(X)$ for any finite subspace X of B_1. With this definition in fact, our amalgamation Lemma 3.2.10 does not work.

3.2.2 Toward an Amalgamation Class

Despite Remark 3.2.7 prevent us from plainly recasting the proof of Lemma 2.2.18 (and Lemma 2.1.11) for $\mathfrak{L}^3_p$-objects, various attempts were made to prove that a non-negative lower bound for δ_3 is preserved under the $\mathfrak{L}^3_p$-amalgamation (3.35).

The strategy is to search (locally) for a free composition at level of $\mathfrak{L}_p^2$-algebras inside the $\mathfrak{L}_p^3$-amalgam. This allows to apply Lemma 3.2.9 and hence obtain submodularity for ∂_3^M. The same procedure could help also to decide whether $\leqslant_3$ is transitive.

The above ideas will be discussed at the end of the section, but first we need to compare δ_3 and ∂_3^M.

To this end we prove the following crucial result.

Corollary 3.2.12. *For any $M \in \mathfrak{L}_p^3$ such that $M_* \in \widetilde{\mathcal{K}}_2$ and any finite $\mathfrak{L}_p^3$-subalgebra $B \subseteq M$ we have $\partial_3^M(B) \leq \delta_3(B)$.*

The proof of the statement above relies on the following result, which will be proved after Lemma 3.2.14 below.

Theorem 3.2.13. *Let $A \supseteq B$ be an extension of finite $\mathfrak{L}_p^2$-algebras, for A in $\mathcal{K}_2$. Assume $A = ssc^A(B)$, then $\dim_{\mathbb{Z}_p}(\ker(\gamma_B^A)) \leq -\delta_2(A/B)$.*

Remark that the above statement doesn't replace Proposition 3.1.6.

Proof of Corollary 3.2.12. Consider the map $\gamma_B^M\colon F_{B_*} \to F_{M_*}$ defined above.

Since $\pi_B = \gamma_B^M \pi_M$ by Lemma 3.2.3, now γ_B^M maps $\ker(\pi_B) = R^3(B)$ onto $\ker(\pi_M) \cap \gamma_B^M(F_B) = R_M^3(B_1)$ with kernel $\ker(\gamma_B^M)$ $(\subseteq R^3(B))$.

Therefore $\dim_{\mathbb{Z}_p}(R_M^3(B_1)) = \dim_{\mathbb{Z}_p}(R^3(B)) - \dim_{\mathbb{Z}_p}(\ker(\gamma_B^M))$ and if A_1 is the self-sufficient closure of B_1 in M_*, then by Corollary 3.2.2 and Remark 2.1.15, we have $\delta_3(B) - \partial_3^M(B) = \delta_2(B) - d_2^M(B) - \dim_{\mathbb{Z}_p}(\ker(\gamma_B^M)) = \delta_2(B) - d_2^A(B) - \dim_{\mathbb{Z}_p}(\ker(\gamma_B^A))$.

By theorem 3.2.13 $\delta_2(B) - d_2^A(B) - k_B^A \geq 0$

□

Abusing our previous notation we denote again by $\gamma_B^A\colon F_B \to F_A$ the canonical map of the free-lift of any extension $A \supseteq B$ of $\mathfrak{L}_p^2$-algebras. The theorem above is based on the following lemma.

Lemma 3.2.14. *Let A be an extension of a finite $\mathfrak{L}_p^2$-algebra C with $A_1 = \langle C_1, a\rangle$ for some a in A_1 linearly independent over C_1.*

Assume $\delta_2(a/C) \leq 0$ and $(\psi_1, \dots, \psi_n)$ is a basis for $R^2(A)$ over $R^2(C)$, where $\psi_i = [c_i, a] - w_i$ for linearly independent $c_1, \dots, c_n$ in C_1 and w_i in $\bigwedge^2 C_1$, then $\dim_{\mathbb{Z}_p}(\ker(\gamma_C^A)) \leq \dim_{\mathbb{Z}_p} R^2(\langle c_1, \dots, c_n\rangle)$.

Proof. We first claim that a basis for $R^2(A/C)$ like in the statement of the lemma can always be found. For assume $A_1 = \langle a, C_1\rangle_{\mathbb{Z}_p}$, if $(\psi_i)_{i=1}^n$ is a basis of $R^2(a/C)$ then, by bilinearity of the Lie product we can assume[4] each ψ_i to be a sum $[c_i, a] + w_i$, where c_i is in C_1 and $w_i \in \bigwedge^2 C_1$.

Since the ψ's are linearly independent over $\bigwedge^2 C_1$, then $c_1, \dots, c_n$ are in particular linearly independent in C_1.

[4] express each ψ_i in basic monomials with respect to any basis of A_1, which completes $\{a\}$ and collect all terms which contain a.

We now arrange a basis $\mathcal{B}$ of A_1 as follows: $\mathcal{B} = \{c_1 > \cdots > c_n > \mathcal{C} > a\}$, where $\mathcal{C}$ is some ordering of a base completion of $c_1, \ldots, c_m$ to C_1.

Recall that $\ker(\gamma_C^A)$ is $(L^3(C_1) \cap \mathbb{J}(A))/\mathbb{J}(C)$ and take an homogeneous element Ψ in $L^3(C_1) \cap \mathbb{J}(A)$ of weight 3.

Since $\mathbb{J}(A) = [R^2(A), A_1] = \sum_{i=1}^n [\psi_i, A_1] + [R^2(C), a] + \mathbb{J}(C)$ we may assume Ψ is a finite homogeneous sum of weight 3:

$$\Psi = \sum_{u \in \mathcal{B}\setminus\{a\},\, i=1}^{n} \lambda_{i,u}[\psi_i, u] + \sum_{i=1}^{n} \theta_i[\psi_i, a] + [\nu, a] \tag{3.40}$$

for $u \in \mathcal{B}$, $\lambda_{u,i}, \theta_i \in \mathbb{Z}_p$ and some $\nu \in R^2(C)$.

We proceed with the same arguments adopted in Proposition 3.1.6 which involved basic and *pre*-basic commutators of weight 3.

We claim first, that terms $[\psi_i, a]$ do not actually appear in the sum above. Consider the unique expression for $\Psi \in L^3(C_1)$ as sum of basic monomials over $\{c_i, \mathcal{C} | i = 1, \ldots, n\}$. These are chosen according to the linear order on $\mathcal{B}$.

On the other hand, from each $[\psi_i, a]$ we have $[c_i, a, a] + [w_i, a]$ and (Engel) basic monomials like $[c_i, a, a]$ cannot be cleared up from the sum (3.40) – applying Jacobi identities – by any other summand. But this would contrast the fact that Ψ belongs to $L^3(C_1)$.

Hence $\Psi = \sum_{i,u} \lambda_{i,u}[\psi_i, u] + [\nu, a]$. Furthermore we affirm that each base element u above must belong to the c_i's. For assume instead some u is in $\mathcal{C}$, and $[\psi_i, u] = [c_i, a, u] + [w_i, u]$ is a non trivial summand in (3.40). The basic monomial $[c_i, a, u]$ – which cannot appear in the expression over C_1 for Ψ – forces the term $[\nu, a]$ to contain $[c_i, u, a]$ as a summand. This implies *both* basic terms $[c_i, a, u]$ and $[u, a, c_i]$ are to be found in the sum of the $\lambda_{i,u}[\psi_i, u]$'s in (3.40), which is impossible if u differs from all c_i's.

With similar arguments follows that ν also actually belongs to $\bigwedge^2 \langle c_1, \ldots, c_m \rangle_{\mathbb{Z}_p}$ and hence to $R^2(\langle c_1, \ldots, c_m \rangle_{\mathbb{Z}_p})$.

To conclude, let k be $\dim_{\mathbb{Z}_p}(\ker(\gamma_C^A))$ and $\{\overline{\Psi}^t\}_{t<k}$ a basis of such kernel, where

$$\Psi^t = \sum_{r,s=1}^{m} \lambda_{r,s}^t [\psi_r, c_s] + [\nu^t, a] \in L^3(C_1) \cap \mathbb{J}(A)$$

for ν^t in $R^2(c_1, \ldots, c_m)$.

After the due simplifications, each element Ψ^t above reduces to

$$\Psi^t = \sum_{r,s=1}^{m} \lambda_{r,s}^t [w_r, c_s]. \tag{3.41}$$

We prove the Lemma by showing linear independence of the $\nu^{t's}$. So assume there are $(\theta_t)_{t<k} \subseteq \mathbb{Z}_p$, such that $\sum_t \theta_t \nu^t = \mathbf{0}$.

It follows $\sum_t \theta_t \Psi^t = \sum_{r,s,t} \lambda^t_{r,s} \theta_t [\psi_r, c_s] = \sum_{r,s=1}^m (\sum_{t=1}^k \theta_t \lambda^t_{r,s})[\psi_r, c_s]$.
But this yields that the sum

$$\sum_{r,s=1}^{m} (\sum_{t=1}^{k} \theta_t \lambda^t_{r,s})[c_r, a, c_s]$$

belongs to $L^3(C_1)$. This is impossible – all $[c_r, a, c_s]$ are basic monomials which are linearly independent over $L^3(C_1)$ – unless $\sum_{t=1}^k \theta_t \lambda^t_{r,s}$ is trivial for all choices of different r, s.

But this gives, then $\sum \theta_t \Psi^t = \mathbf{0}$ and hence θ^t has to be trivial for all $t < k$. It follows $k \leq \dim_{\mathbb{Z}_p}(R^2(\langle c_1, \dots, c_m \rangle_{\mathbb{Z}_p}))$ as desired.

□

In the sequel we denote by K^A_B the kernel of $\gamma^A_B \colon F_B \to F_A$ for $A \supseteq B \in \mathfrak{L}^2_p$ and we set $k^A_B := \dim_{\mathbb{Z}_p}(K^A_B)$.

For any two extensions $A \supseteq B \supseteq C$, since $\gamma^A_C = \gamma^B_C \gamma^A_B$, γ^C_B maps K^A_B into K^A_C with kernel K^C_B. In particular we have

$$k^A_C \leq k^B_C + k^A_B. \tag{3.42}$$

Proof of Theorem 3.2.13. We prove the statement by induction on $l = \dim_{\mathbb{Z}_p}(A_1/B_1)$. For $l = 0$ there is nothing to prove.

For $l = 1$ remark that B is *not* strong in A and apply Lemma 3.2.14 to the finite extension $A \supseteq B$. This gives $k^A_B \leq \dim_{\mathbb{Z}_p}(R^2(c_1, \dots, c_n))$ where $c_1, \dots, c_n$ are linearly independent elements of B_1 and $n = \dim_{\mathbb{Z}_p}(R^2(A/B)) > 1$. Now since $A \models \Sigma^2(2)$ then $\dim_{\mathbb{Z}_p}(R^2(c_1, \dots, c_n)) < n$ and we have $k^A_B \leq n - 1 = -\delta_2(A/B)$.

Assume $l \geq 2$. We divide the proof into different cases:

Case 1: *There exists a proper subspace H_1 of A_1 such that $B_1 \subsetneq H_1 \subsetneq A_1$ and $\delta_2(H) \leq \delta_2(B)$.*

The properties of the self-sufficient closure imply $\delta_2(H) > \delta_2(A)$ and $A = ssc^A(H)$. Take such an H which is minimal with respect to inclusion and with minimal $\delta_2(H)$.

Case 1.1: $\delta_2(H) = \delta_2(B)$.

By the choice of H, $B \leqslant_2 H$, hence $k^H_B = 0$ and $k^A_B \leq k^A_H$. By induction now $k^A_H \leq -\delta_2(A/H) = -\delta_2(A/B)$. And the assertion follows.

Case 1.2: $\delta_2(H) < \delta_2(B)$.

In this case we have $H = ssc^H(B)$ and $A = ssc^A(H)$. By applying the inductive hypothesis we obtain $k^A_B \leq k^H_B + k^A_H \leq -\delta_2(H/B) - \delta_2(A/H) = -\delta_2(A/B)$.

Case 2: *There is no such H_1 like in Case 1. That is for all $B_1 \subsetneq H_1 \subsetneq A_1$ we have $\delta_2(B) < \delta_2(H)$.*

Take a subspace $C_1 \supseteq B_1$ with codimension 1 in A_1, such that $A_1 = \langle C_1, a\rangle$ for some a in A_1. We have $B \leqslant_2 C$ and hence γ_B^C is mono.

We proceed like in Lemma 3.2.14 to find a basis

$$\psi_i = [c_i, a] + w_i \quad i = 1, \dots, n \tag{3.43}$$

of $R^2(A)$ over $R^2(C)$ where $w_i \in \bigwedge^2 C_1$ and the set $(c_1, \dots, c_n) \subseteq C_1$ is linearly independent. Also $n > 1$ since $\delta(A/C) < 0$.

Moreover, any element Ψ of K_C^A is the image modulo $\mathbb{J}(C)$ of a sum

$$\Psi = \sum_{i,j=1}^{n} \lambda_{i,j}[\psi_i, c_j] + [\nu, a] = \sum_{i,j=1}^{n} \lambda_{i,j}[w_i, c_j] \tag{3.44}$$

for some ν in $R^2(c_1, \dots, c_n)$ (cfr. (3.41) above).

Case 2.1: *C_1 is generated by B_1 and the $c_1, \dots, c_n$.*

For a suitable choice of m independent elements $b_1, \dots, b_m$ of B_1 and $n - m =: h$ elements $a_1, \dots, a_h$ of C_1 independent over B_1, we may assume that $c_i = b_i$ for $i = 1, \dots, m$ and that $c_{m+i} = a_i$ for $i = 1, \dots, h$.

We arrange and order a basis of A_1 by taking

$$\{\mathcal{B} > b_1 > \cdots > b_m > a_1 > \cdots > a_h > a\}$$

where $\mathcal{B}$ is a basis completion of $\{b_i \mid i = 1, \dots, m\}$ to a basis of B_1 and $C_1 = \langle B_1, a_j \mid j = 1, \dots, h\rangle_{\mathbb{Z}_p}$.

Observe also that we may assume $m \geq 1$, for otherwise by comparing the expression in (3.44), we would have $L^3(B_1) \cap \mathbb{J}(A) = \mathbf{0}$ and hence

$$K_B^A \simeq \gamma_B^C(K_B^A) = K_C^A \cap \gamma_B^C(F_B) \simeq \frac{(L^3(B_1) \cap \mathbb{J}(A)) + \mathbb{J}(C)}{\mathbb{J}(C)} = \mathbf{0}$$

and the result would trivially follow.

If k denotes the dimension of $R^2(\langle b_i, a_j\rangle_{\mathbb{Z}_p})$ and we set $k_b = \dim_{\mathbb{Z}_p}(R^2(\langle b_i\rangle_{\mathbb{Z}_p}))$, then we have $k - k_b \leq \dim_{\mathbb{Z}_p}(R^2(C/B))$ and as $\dim_{\mathbb{Z}_p}(R^2(a/C)) = n = m + h$,

$$\begin{aligned} -\delta(A/B) &= \dim_{\mathbb{Z}_p}(R^2(a/C)) + \dim_{\mathbb{Z}_p}(R^2(C/B)) - (h+1) \geq \\ &\geq m + h + k - k_b - h - 1 \geq m - 1 - k_b + k \end{aligned}$$

Now since B has $\Sigma^2(2)$, $m - 1 - k_b \geq 0$ and hence $k \leq -\delta(A/B)$.

Since $B \leqslant_2 C$ we have $k_B^A \leq k_C^A$ while by Lemma 3.2.14 we have $k_C^A \leq k$ and hence $k_B^A \leq -\delta(A/B)$ follows.

Case 2.2: *C_1 is not generated by the c_i's over B_1 only.*

In this case, me may assume C_1 has an ordered basis

$$\mathcal{C} = \{\mathcal{B} > b_1 > \cdots > b_m > a_1 > \cdots > a_h > e_1 > \cdots > e_r\}$$

where $r \geq 1$, the set $\{b_i, a_j \mid i = 1, \dots, m,\ j = 1, \dots, h\}$ play the role of $\{c_i \mid i = 1, \dots, n\}$ as before, and $\mathcal{B}$ completes $\{b_1, \dots, b_m\}$ to a basis of B_1. Also let a, which completes $\mathcal{C}$ to a basis of A_1, be smaller than any other element.

As previously observed, being $K_B^A \simeq \gamma_B^C(K_B^A) = (L^3(B_1) \cap \mathbb{J}(A)) + \mathbb{J}(C)/\mathbb{J}(C)$, we take into account the representative Ψ in $L^3(B_1) \cap \mathbb{J}(A)$ for some arbitrary element $\overline{\Psi}$ in $\gamma_B^C(K_B^A)$.

For all j, if we set $\hat{\psi}_j = \sum_i \lambda_{i,j}\psi_i$ and $\hat{w}_j = \sum_i \lambda_{i,j} w_i$, expression (3.44) for Ψ becomes

$$\Psi = \sum_j [\hat{\psi}_j, c_j] + [\nu, a] = \sum_j [\hat{w}_j, c_j].$$

Moreover we can replace each $\hat{w}_j$ with a sum of basic monomials $\sum_\alpha s_j^\alpha [x^{j,\alpha}, y^{j,\alpha}]$ with $x^{j,\alpha} > y^{j,\alpha}$ in $\mathcal{C}$. We obtain in particular

$$\Psi = \sum_{\alpha,j} s_j^\alpha [x^{j,\alpha}, y^{j,\alpha}, c_j]. \tag{3.45}$$

which has to be compared with the unique expression of Ψ in basic commutators over $\mathcal{B} \cup \{b_i \mid i = 1, \dots, m\}$.

Consider the subspace $C' = \langle B_1, a_1, \dots, a_h, e_1, \dots, e_{r-1}, a\rangle_{\mathbb{Z}_p}$, we claim that $\hat{w}_j$ belongs to $\bigwedge^2 C_1'$ for all j. If this is not the case, then a term $[x, e_r]$ appears *with a nontrivial coefficient* from $\mathbb{Z}_p$, in the sum presenting $\hat{w}_j$ for some j. This implies that the weight 3 commutator $[x, e_r, c_j]$ appears in (3.45) with a non-zero coefficient. Also remark that $[x, e_r, c_j]$ is basic, because $x > e_r < c_j$.

Since e_r does not appear among the c_i's, this basic commutator can in no way arise[5] from – nor be eliminated by – a prebasic monomial, that is, by a term $[u, v, e_r]$ in (3.45) with $u > v > e_r$. Since Ψ is in $L^3(B_1)$, the claim above follows.

Now if $\hat{w}_j$ is in $\bigwedge^2 C_1'$, then in particular $\hat{\psi}_j$ is in $R^2(C')$ and $[\hat{\psi}_j, c_j]$ belongs to $\mathbb{J}(C')$ for all j. Since ν belongs to $R^2(\langle c_1, \dots, c_n\rangle_{\mathbb{Z}_p})$, then Ψ is in $\mathbb{J}(C')$.

On the other hand, as we are in "Case 2" we have $B \leqslant_2 C'$ and hence $L^3(B_1) \cap \mathbb{J}(C') = \mathbb{J}(B)$, but then Ψ belongs to $\mathbb{J}(B) \subseteq \mathbb{J}(C)$, and $\overline{\Psi}$ is trivial. The statement of the theorem is (trivially) true in this case as well.

□

In parallel with axiom $\Sigma^2(2)$, now define for M in $\mathfrak{L}_p^3$ a denumerable set of $\mathscr{L}^3$-sentences $\Sigma^3(2)$ which express:

($\Sigma^3(2)$) for any finite $A_1 \subseteq M_1$, $\delta_3(A) \geq 0$[6]

[5] by means of Jacobi identities. Cfr. the proof of Proposition 3.1.6.

[6] the weak bound ≥ 0 will have to be replaced with some stronger property, similar to that required by $\Sigma^2(2)$ from Chapter 2.

A natural candidate for a suitable amalgamation class among $\mathfrak{L}_p^3$-algebras with $\Sigma^3(2)$ has to be found within – possibly a subset of – the family

$$\widetilde{\mathcal{K}}_3 = \{M \in \mathfrak{L}_p^3 \mid M_* \in \widetilde{\mathcal{K}}_2, M \models \Sigma^3(2)\}.$$

Remark 3.2.15. It is the same whether we check $\Sigma^3(2)$ of an $\mathfrak{L}_p^3$-algebra M with δ_3 or with ∂_3^M.

In fact Corollary 3.2.12 ensures $\delta_3(A) \geq \partial_3^M(A)$ for any finite $A_1 \subseteq M_1$, while on the other hand, $\partial_3^M(A) \geq \partial_3^M(ssc(A)) = \delta_3(ssc(A))$. In other words M belongs to $\widetilde{\mathcal{K}}_3$ exactly if $M_* \in \widetilde{\mathcal{K}}_2$ and $\partial_3^M(A) \geq 0$ for all finite $A_1 \subseteq M_1$.

In order to study (AP) within $\widetilde{\mathcal{K}}_3$, we first find a setting in which the free amalgamation (3.35) inherits property $\Sigma^3(2)$ of its constituents.

Assume N, A, B are $\mathfrak{L}_p^3$-algebras in $\widetilde{\mathcal{K}}_3$ and suppose $N \supseteq B \leqslant_3 A$. If we take the amalgam M like in (3.35) we have $N \leqslant_3 M \supseteq A$.

We also assume that the free $\mathfrak{L}_p^2$-amalgam $M_* = N_* \circledast_{B_*} A_*$ is in $\widetilde{\mathcal{K}}_2$, which is not such a serious restriction. Indeed a variant to the class defined above could rely on $\mathfrak{L}_p^3$-algebras M such that M_* is a *self-sufficient and algebraically closed* subalgebra of $\mathbb{K}$, where $\mathbb{K} \in \widetilde{\mathcal{K}}_2$ is the Fraïssé limit of the class $\mathcal{K}_2$ defined in the previous chapter 2 (cfr. Proposition 2.2.18 and Lemma 2.3.8).

Lemma 3.2.16. *Assume $M \in \mathfrak{L}_p^3$ is the amalgam above, for $N \supseteq B \leqslant_3 A$ in $\widetilde{\mathcal{K}}_3$ with $M_* = N_* \circledast_{B_*} A_* \in \widetilde{\mathcal{K}}_2$. Assume E_1 is a finite subspace of M_1, let C_1 denote $N_1 + E_1$ and set as usual C to be $\langle C_1 \rangle^M$. Suppose C_* is the $\mathfrak{L}_p^2$-free amalgam of N_* and E_* over $\langle N_1 \cap E_1 \rangle^{C_*}$, then $\delta_3(E) \geq 0$.*

Proof. First notice $\delta_3(E) \geq \partial_3^C(E)$ by Lemma 3.2.12.

Now with the above assumptions, Lemma 3.2.9 and (3.25) yield

$$R_C^3(E_1) \cap R_C^3(N_1) = R_C^3(E_1 \cap N_1). \tag{3.46}$$

In the same way submodularity (2.5) on page 37 was obtained for $\mathfrak{L}_p^2$, now (3.46) implies $\partial_3^C(E) \geq \partial_3^C(E/N) + \partial_3^C(E_1 \cap N_1)$.

Since N is δ_2-strong in C, by Lemma 3.2.6 and Remark 3.2.15 we have $\partial_3^C(E_1 \cap N_1) = \partial_3^N(E_1 \cap N_1) \geq 0$ for $N \in \widetilde{\mathcal{K}}_3$.

We have to prove $\partial_3^C(E/N) \geq 0$. To achieve this we will show $\partial_3^C(E/N) \geq \partial_3^M(E/N)$ and use the fact $N \leqslant_3 M$.

Notice that by the definition of C and since $N \leqslant_2 M$ we have $d_2^C(E/N) = \delta_2(E/N)$ and by Definition 3.2.4 $R_C^3(E/N) = R^3(C)/R^3(N)$.

Since $ssc(C_1)$ is finite over N_1 and $R_M^3(C_1) = \gamma_C^M(R^3(C))$, we obtain $\partial_3^C(E/N) - \partial_3^M(E/N) = -\delta_2(ssc^M(C)/C) - \dim_{\mathbb{Z}_p}(K_C^M)$, where as above $K_C^M = K_C^{ssc(C)}$ is the kernel of γ_C^M.

Now by the finite character of ssc^M described in Proposition 2.1.17, Theorem 3.2.13 applies with minor changes to the present situation and therefore the dimension of K_C^M is at most $-\delta_2(ssc^M(C)/C)$.

□

Notice that the subalgebra C_* is indeed a free amalgam of N_* and $\langle C_1 \cap A_1 \rangle^{C_*}$, but the proof actually needs the stronger assumptions stated above.

Suppose M is the above $\mathfrak{L}_p^3$-amalgam of N and A over B. As a last remark to try solving the amalgamation issue inside $\widetilde{\mathcal{K}}_3$ we can address to the following problem.

Remark. M amalgamates N and A over B, for A, B and N in $\widetilde{\mathcal{K}}_3$ as above. Assume $M_* = N_* \circledast_{B_*} A_*$ is in $\widetilde{\mathcal{K}}_2$ and for each finite E_1 of M_1, there is a subspace $\widetilde{E}_1 \supseteq E_1$ with the features:

- $d_2(\widetilde{E}) = d_2(E)$
- $N_* + \widetilde{E}_*$ is the free amalgam of N_* and $\widetilde{E}_*$ over $\langle N_1 \cap \widetilde{E}_1 \rangle^{N_* + \widetilde{E}_*}$.

Then M lay in $\widetilde{\mathcal{K}}_3$.

This confirms, it could be useful to work with algebraically closed underlying $\mathfrak{L}_p^2$-structures, in the sense of Lemma 2.3.8.

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Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 30. Juni 2011 Andrea Amantini